Einstein's comprehensive 1907 essay on relativity, part III

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This is the concluding part of the English rendition of Einstein's 1907 essay on relativity, of which part I appeared in the June 1977 issue of this Journal and part II in the September 1977 issue. It consists of a direct translation of the last part of the essay, part V, entitled "Principle of Relativity and Gravitation," and of a few added footnotes.

A. INTRODUCTION

Einstein's 1907 essay on relativity[1] does not appear to be widely known. Yet, as noted in the Introduction to the first part of the present English rendition of this essay,[2] it is of substantial interest both on didactic and historic grounds. Its didactic value, relating to the treatment of a number of basic topics in Special Relativity, is particularly in evidence in the portion of Einstein's essay that is dealt with in the second part of the present rendition.[3] Its historical importance is associated mainly with the genesis of special relativity, and also with the genesis of general relativity. Part V, the last part of Einstein's 1907 essay, contains Einstein's first published expression of his initial highly important seminal ideas on the latter subject. It is translated here, as far as seemed feasible, verbatim, with a few added mainly explanatory notes.

It may perhaps not be amiss to point out here that the latter notes, as well as those presented in the other two parts of the present rendition, and in the partial translation of Einstein's first paper on relativity,[4] have for their principal aim only the facilitating of a close reading of the respective fundamental papers of Einstein, whether historically or pedagogically motivated.[5]

As in the previous parts of this rendition all the original footnotes are labeled by lower-case roman letters, and the added footnotes by arabic numerals.

B. TRANSLATION OF THE GRAVITATIONAL PART OF EINSTEIN'S 1907 MEMOIR

V. Principle of Relativity and Gravitation

17. Accelerated Reference System and Gravitational Field

Until now we have applied the principle of relativity—i.e., the assumption that the laws of nature are independent of the state of motion of the reference system—only to *nonaccelerated* reference systems. Is it conceivable that the principle of relativity holds also for systems which are accelerated with respect to each other?

This is not really the place for the exhaustive treatment of this subject. Since it forces itself, however, on the mind of anyone who has followed the previous applications of the principle of relativity, I shall not refrain here from taking a position on the question.

We consider two systems of motion, Σ_1 and Σ_2. Suppose Σ_1 is accelerated in the direction of its X axis, and γ is the magnitude (constant in time) of this acceleration. Suppose Σ_2 is at rest,[6] but situated in a homogeneous gravitational field, which imparts to all objects an acceleration $-\gamma$ in the direction of the X axis.

As far as we know, the physical laws with respect to Σ_1 do not differ from those with respect to Σ_2; this derives from the fact that all bodies are accelerated alike in the gravitational field. We have therefore no reason to suppose in the present state of our experience that the systems Σ_1 and Σ_2 differ in any way, and will therefore assume in what follows the complete physical equivalence of the gravitational field and the corresponding acceleration of the reference system.[7]

This assumption extends the principle of relativity to the case of uniformly accelerated translational motion of the coordinate system. The heuristic value of the assumption lies therein that it makes possible the replacement of a homogeneous gravitational field by a uniformly accelerated reference system, the latter case being amenable to theoretical treatment to a certain degree.

18. Space and Time in a Uniformly Accelerated Reference System

We consider first a body whose individual material points possess relative to the nonaccelerated reference system S, at a fixed time t of S, a certain acceleration but no velocity. What influence does this acceleration γ have on the shape of the body with respect to S?

If such an influence exists, it will consist in a dilatation of constant ratio in the direction of the acceleration, and possibly in the two directions perpendicular to this direction[8]; since an influence of another kind is precluded by considerations of symmetry. Those dilatations arising from the acceleration (if they exist at all) must be even functions of γ; and they can be thus disregarded when one restricts oneself to the case when γ is so small that terms of the second and higher powers in γ may be neglected.[9] Since we wish to confine ourselves in the sequel to this case, we do not have therefore to assume any influence of the acceleration on the shape of a body.

We consider now a reference system Σ which is uniformly accelerated relative to the nonaccelerated reference system S in the direction of the X axis of the latter. Let the clocks and the measuring rod of Σ, when examined at rest, be the same as the clocks and the measuring rod of S. Let the origin of coordinates of Σ move along the X axis of S, and let the axes of Σ remain parallel to those of S. There exists at every instant a nonaccelerated reference system S', whose coordinate axes coincide with the coordinate axes of Σ at that instant (for a fixed time t' of S'). If a point-event occurring at this time t' has the coordinates ξ,η,ζ with respect to Σ, then

$$x' = \xi, \quad y' = \eta, \quad x' = \zeta,$$

since by the foregoing discussion we must not assume any influence of the acceleration upon the shape of the measuring bodies used in measuring ξ,η,ζ. Let us imagine further that at this time t' of S' the clocks of Σ are so adjusted that their reading at this instant is t'. What can we say about the rate of the clocks in the next time element τ?

First we have to bear in mind that a specific influence of the acceleration upon the rate of the clocks does not enter into consideration, since it would have to be of the order of γ^2. Moreover, since we can neglect, as being of order τ^2, the influence upon the rate of the clocks of the velocity attained during τ as well as of the path traveled by the clocks relative to those of S' during the time τ, therefore for the time element τ the readings of the clocks of Σ are fully replaceable[10] by the readings of the clocks of S'.

It follows from the foregoing discussion, that in the time element τ light in vacuum propagates with the universal velocity c relative to Σ, if we define simultaneity in the system S' which is instantaneously at rest relative to Σ, and apply for the measurement of time and lengths, clocks and measuring rods which are the same as those used in the measurement of time and lengths in nonaccelerated systems. The principle of the constancy of the velocity of light can thus be applied also here for the definition of simultaneity, provided one confines oneself to very small light paths.

We imagine now that the clocks of Σ are set in the indicated manner at that time $t = 0$ of S when Σ is momentarily at rest relative to S. The totality of the readings of the clocks of Σ so set, shall be called the "local time" σ of the system Σ. As one recognizes immediately, the physical significance of the local time σ is as follows. If one utilizes this local time σ for the temporal labeling of processes occurring at individual space elements of Σ, then the laws obeyed by those processes cannot depend on the position of the particular spatial element, i.e., on its coordinates, if one employs at the different spatial elements not only the same clocks, but the same measuring devices as well.

On the other hand, we must not consider the local time σ as simply the "time" of Σ, because, in fact, two events taking place at two different points of Σ are not simultaneous in the sense of the above definition when their local times σ are equal to each other. Since, namely, any two clocks of Σ are synchronous with respect to S at the time $t = 0$, and undergo the same motion, they remain continuously synchronous with respect to S. On this account, according to Sec. 4,[11] they are not synchronous with respect to a reference system S' that is momentarily at rest relative to Σ and in motion relative to S, and therefore, according to our definition, neither are they synchronous with respect to Σ.

We define now the "time" τ of the system Σ as the totality of those readings of the clock located at the origin of coordinates of Σ, which are simultaneous, in the sense of the above definition, with the events to be temporally labelled.[a]

We will now find the relationship obtaining between the time τ and the local time σ of an event. From the first of Eqs. (1)[11] it follows that two events are simultaneous with respect to S', and hence also with respect to Σ, when

$$t_1 - (vx_1/c^2) = t_2 - (vx_2/c^2),$$

where the indices refer to the one and to the other point-event, respectively. We restrict ourselves at first to the consideration of such short times,[b] that all terms containing second or higher powers of τ or of v can be discarded; then, by reference to Eqs. (1) and (2),[11] we have to set

$$x_2 - x_1 = x_2' - x_1' = \xi_2 - \xi_1;$$
$$t_1 = \sigma_1, \quad t_2 = \sigma_2; \quad v = \gamma t = \gamma\tau,^{12}$$

so that we obtain from the above equations:

$$\sigma_2 - \sigma_1 = \gamma\tau(\xi_2 - \xi_1)/c^2.$$

If we place the first point-event at the origin of coordinates, so that $\sigma_1 = \tau$ and $\xi_1 = 0$, we obtain upon dropping the index for the second point-event,

$$\sigma = \tau[1 + (\gamma\xi/c^2)]. \qquad (30)$$

This equation is valid, to begin with, when τ and ξ lie below certain bounds. It holds obviously for arbitrarily large τ, if the acceleration γ is constant with respect to Σ,[13] because then the connection between σ and τ must be linear. For arbitrarily large ξ Eq. (30) does not hold. In fact, since the choice of the origin of coordinates cannot influence the relation in question, one concludes that Eq. (30) must be replaced in all strictness by the equation

$$\sigma = \tau e^{\gamma\xi/c^2}.$$

We shall, however, retain formula (30).

According to Sec. 17, Eq. (30) is to be applied also to a coordinate system in which a homogeneous gravitational field is acting. In this case we have to set $\Phi = \gamma\xi$, where Φ denotes the gravitational potential, so that we obtain

$$\sigma = \tau[1 + (\Phi/c^2)]. \qquad (30a)$$

We have defined two kinds of time for Σ. Which of the two definitions do we have to utilize in the different cases? Let us suppose that at each of two places of different gravitational potential ($\gamma\xi$) there exists a physical system, and that we wish to compare the physical quantities associated with them. To this end, we shall clearly proceed most naturally as follows. We betake ourselves with our measuring devices first to the one physical system, and carry out our measurements there; and then we betake ourselves with our measuring devices to the other system to carry out here the identical measurements. If the measurements yield the identical results in both places, we shall designate the two physical systems as "identical." Among the mentioned measuring devices there exists a clock, with which we measure the local time σ. From this it follows that for the definition of physical quantities at a given place of the gravitational field, we quite naturally utilize the time σ.

But if one deals with a phenomenon that necessitates the simultaneous consideration of objects situated at places of different gravitational potential, then we must employ the time τ in the terms where the time appears explicitly (i.e., not only in the definition of physical quantities); since otherwise the simultaneity of the events would not be expressed by the identity of the values of their time. Since in the definition of the time τ one does not employ an arbitrarily chosen instant, but rather a clock situated at an arbitrarily chosen place, the laws of nature, when one uses the time τ, cannot vary therefore with the time, but may well vary with the place.

19. Influence of the Gravitational Field Upon Clocks

If at a point P of the gravitational field Φ there is situated a clock which indicates the local time, then according to Eq.

(30a) its indications are $1 + (\Phi/c^2)$ greater than the time τ, i.e., it runs $1 + (\Phi/c^2)$ faster than an identically constructed clock situated at the origin of coordinates. Suppose that an observer situated anywhere in space ascertains in some manner the indications of the two clocks, e.g., by optical means. Since the time interval $\Delta\tau$, which elapses between the instant of an indication of one of the clocks and its being perceived by the observer, is independent of τ, the clock at P runs for an observer situated anywhere in space $1 + (\Phi/c^2)$ times faster than the clock at the origin of coordinates. It is in this sense that we can say that the process taking place within the clock—and more generally, every physical process—proceeds at a rate which is the faster the greater the gravitational potential of the place where it occurs.

Now there exist "clocks," which are to be found at places of different gravitational potential and whose rates can be controlled very precisely; these are the generators of spectral lines. It follows from the above discussion[c] that the light coming from the surface of the Sun, which arises from such a generator, possesses a wavelength that is greater by about a two-millionth part than that of the light generated by identical material on the surface of the Earth.

20. Influence of Gravitation Upon Electromagnetic Processes

If we refer an electromagnetic process at a given instant to a nonaccelerated reference system S' that is momentarily at rest relative to the reference system Σ, which is accelerated as above, then by Eqs. (5) and (6)[14] we have the equations

$$\left(\rho' u'_x + \frac{\partial X'}{\partial t'}\right)\Big/ c = \frac{\partial N'}{\partial y'} - \frac{\partial M'}{\partial z'}, \quad \text{etc.,}$$

and

$$\frac{\partial L'}{c\partial t'} = \frac{\partial Y'}{\partial z'} - \frac{\partial Z'}{\partial y'}, \quad \text{etc.}$$

According to the above, we can immediately identify the quantities ρ', u', X', L', x', etc., referred to S' with the corresponding quantities ρ, u, X, L, ξ, etc., referred to Σ, as long as we confine ourselves to an infinitely short time,[d] which is infinitely close to the time of relative rest of S' and Σ. Moreover, we have to replace t' by the local time σ. However, we may not simply set

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial \sigma},$$

because a point at rest with respect to Σ, to which the equations transformed to Σ are to be referred, changes its velocity relative to S' during the time element $dt' = d\sigma$. To this change there corresponds according to Eqs. (7a) and (7b)[14] a temporal change in the field components which are referred to Σ. We have therefore to set

$$\frac{\partial X'}{\partial t'} = \frac{\partial X}{\partial \sigma}, \quad \frac{\partial Y'}{\partial t'} = \frac{\partial Y}{\partial \sigma} + \frac{\gamma N}{c}, \quad \frac{\partial Z'}{\partial t'} = \frac{\partial Z}{\partial \sigma} - \frac{\gamma M}{c},$$
$$\frac{\partial L'}{\partial t'} = \frac{\partial L}{\partial \sigma}, \quad \frac{\partial M'}{\partial t'} = \frac{\partial M}{\partial \sigma} - \frac{\gamma Z}{c}, \quad \frac{\partial N'}{\partial t'} = \frac{\partial N}{\partial \sigma} + \frac{\gamma Y}{c}.$$

The electromagnetic equations referred to Σ read thus, to begin with,

$$\left(\rho u_\xi + \frac{\partial X}{\partial \sigma}\right)\Big/ c = \frac{\partial N}{\partial \eta} - \frac{\partial M}{\partial \zeta}$$
$$\left(\rho u_\eta + \frac{\partial Y}{\partial \sigma} + \frac{\gamma N}{c}\right)\Big/ c = \frac{\partial L}{\partial \zeta} - \frac{\partial N}{\partial \xi},$$
$$\left(\rho u_\zeta + \frac{\partial Z}{\partial \sigma} - \frac{\gamma M}{c}\right)\Big/ c = \frac{\partial M}{\partial \xi} - \frac{\partial L}{\partial \eta},$$
$$\frac{\partial L}{c\partial \sigma} = \frac{\partial Y}{\zeta \xi} - \frac{\partial Z}{\partial \eta},$$
$$\left(\frac{\partial M}{\partial \sigma} - \frac{\gamma Z}{c}\right)\Big/ c = \frac{\partial Z}{\partial \xi} - \frac{\partial X}{\partial \zeta},$$
$$\left(\frac{\partial N}{\partial \sigma} + \frac{\gamma Y}{c}\right)\Big/ c = \frac{\partial X}{\partial \eta} - \frac{\partial Y}{\partial \xi}.$$

We multiply these equations by $1 + (\gamma\xi/c^2)$, and set for short

$$X^* = X[1 + (\gamma\xi/c^2)], \quad Y^* = Y[1 + (\gamma\xi/c^2)], \quad \text{etc.,}$$
$$\rho^* = \rho[1 + (\gamma\xi/c^2)].$$

Upon neglecting terms of second degree in γ, we obtain then the equations

$$\left(\rho^* u_\xi + \frac{\partial X^*}{\partial \sigma}\right)\Big/ c = \frac{\partial N^*}{\partial \eta} - \frac{\partial M^*}{\partial \zeta},$$
$$\left(\rho^* u_\eta + \frac{\partial Y^*}{\partial \sigma}\right)\Big/ c = \frac{\partial L^*}{\partial \zeta} - \frac{\partial N^*}{\partial \xi}, \qquad (31a)$$
$$\left(\rho^* u_\zeta + \frac{\partial Z^*}{\partial \sigma}\right)\Big/ c = \frac{\partial M^*}{\partial \xi} - \frac{\partial L^*}{\partial \eta}.$$

$$\frac{\partial L^*}{c\partial \sigma} = \frac{\partial Y^*}{\partial \zeta} - \frac{\partial Z^*}{\partial \eta},$$
$$\frac{\partial M^*}{c\partial \sigma} = \frac{\partial Z^*}{\partial \omega} - \frac{\partial X^*}{\partial \zeta}, \qquad (32a)$$
$$\frac{\partial N^*}{c\partial \sigma} = \frac{\partial X^*}{\partial \eta} - \frac{\partial Y^*}{\partial \xi}.$$

From these equations one sees first how the gravitational field influences static and stationary phenomena. The regularities that are obtained are the same as in the gravitation-free field, except for the substitution of $X[1 + (\gamma\xi/c^2)]$, etc., for X, etc., and $\rho[1 + (\gamma\xi/c^2)]$ for ρ.

Moreover, in order to survey the course of nonstationary states we employ the time τ for terms that involve differentiation with respect to time, as well as for the definition of the velocity of electricity, i.e., we set according to Eq. (30),

$$\frac{\partial}{\partial \tau} = \left(1 + \frac{\gamma\xi}{c^2}\right)\frac{\partial}{\partial \sigma}$$

and

$$w_\xi = [1 + (\gamma\xi/c^2)]u_\xi.$$

We obtain thus

$$\left(\rho^* w_\xi + \frac{\partial X^*}{\partial \tau}\right)\Big/ c[1 + (\gamma\xi/c^2)] = \frac{\partial N^*}{\partial \eta} - \frac{\partial M^*}{\partial \zeta}, \text{ etc.,} \qquad (31b)$$

and

$$\left(\frac{\partial L^*}{\partial \tau}\right)\Big/ c\,[1 + (\gamma\xi/c^2)] = \frac{\partial Y^*}{\partial \zeta} - \frac{\partial Z^*}{\partial \eta}, \text{ etc.} \qquad (32b)$$

These equations, too, are of the same form as the corresponding ones in the acceleration-free or gravitation-free space, but here instead of c there appears the quantity

$$c[1 + (\gamma\xi/c^2)] = c[1 + (\Phi/c^2)].$$

It follows from this that the light rays that are not propagated in the direction of the ξ axis are bent by the gravitational field; as is easily seen, the change in direction per centimeter of light path amounts to $(\gamma/c^2)\sin\phi$, where ϕ is the angle between the direction of the gravitational force and that of the light ray.

By means of these equations and those which connect field strengths and electric currents at a given place according to the theory of optics of stationary bodies, it is possible to ascertain the influence of the gravitational field upon optical phenomena for stationary bodies. It should be borne in mind in this connection that those equations from the optics of stationary bodies are valid for the local time σ. Unfortunately, the influence of the Earth's gravitational field is according to our theory so slight (because of the smallness of $\gamma x/c^2$), that there exists no prospect for a comparison of the results of the theory with experience.

If we multiply Eqs. (31a) and (32a) successively by $X^*/4\pi, \cdots, N^*/4\pi$ and integrate over infinite [i.e., all of] space, we obtain, using our previous notation:

$$\int [1 + (\gamma\xi/c^2)]^2(\rho/4\pi)(u_\xi X + u_\eta Y + u_\zeta Z)d\omega + \int [1 + (\gamma\xi/c^2)]^2(1/8\pi) \times \frac{\partial(X^2 + Y^2 + \cdots + N^2)}{\partial\sigma} d\omega = 0.$$

$\rho(u_\xi X + u_\eta Y + u_\zeta Z)/4\pi$ is the energy η_σ conveyed to the matter per unit volume and unit local time σ; when this energy is measured by means of the measuring devices located at the place in question. Hence by Eq. (30), $\eta_\tau = \eta_\sigma[(1 + (\gamma\xi/c^2)]$ is the energy (similarly measured) conveyed to the matter per unit volume and per unit of time τ. $(X^2 + Y^2 + \cdots + N^2)/8\pi$ is the electromagnetic energy ϵ per unit volume—similarly measured. If we bear in mind, further, that according to Eq. (30) we have to set $\partial/\partial\sigma = [1 - (\gamma\xi/c^2)]\partial/\partial\tau$, we obtain

$$\int [1 + (\gamma\xi/c^2)]\eta_\tau\, d\omega + \frac{d}{d\tau}\int [1 + (\gamma\xi/c^2)]\epsilon\, d\omega = 0.$$

This equation expresses the principle of the conservation of energy and contains a very remarkable result. An energy or a transport of energy which has the respective value $E = \epsilon\, d\omega$ or $E = \eta\, d\omega\, d\tau$, when measured at a given spot, contributes to the energy integral in addition to the value E corresponding to its quantity, also a value $E\gamma\xi/c^2 = E\Phi/c^2$ corresponding to its position. To every energy E there belongs thus in the gravitational field an energy of position, which is as large as the energy of position of a "ponderable" mass of magnitude E/c^2.

The law deduced in Sec. 11,[14] that to a quantity of energy E there belongs a mass of magnitude E/c^2, holds thus not only for the inertial, but also for the gravitational mass, provided the assumption introduced in Sec. 17 is valid.

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[a]The symbol "τ" is thus employed here in a different sense than previously.

[b]Thereby there arises also, according to (1),[11] a certain restriction regarding the values of $\xi = x'$.

[c]By assuming that Eq. (30a) holds also for a nonhomogeneous gravitational field.

[d]This restriction does not impair the domain of validity of our results, since by the nature of things, the laws to be derived cannot depend on the time.

[1]A. Einstein, Jahrb. Radioakt. Elektronik **4**, 411 (1907). Corrections by Einstein in Jahrb. Radioakt. Elektronik **5**, 98 (1908).

[2]H. M. Schwartz, Am. J. Phys., **45**, 512 (1977).

[3]H. M. Schwartz, Am. J. Phys. **45**, 811 (1977).

[4] H. M. Schwartz, Am. J. Phys. **45**, 18 (1977).

[5]The work contained in Refs. 2–4, as well as in the author's rendition of Poincaré's Rendiconti paper on relativity that appeared in this journal [**39**, 1287 (1971); **40**, 862 (1972); **40**, 1282 (1972)], were in fact undertaken after the initiation of an historical study, entitled *Lorentz, Poincaré, Einstein, and the Special Theory of Relativity* (that it is planned to complete shortly), which pointed out a need for readier access to the above-mentioned works of Poincaré and of Einstein on special relativity. Similarly, the present paper is in part motivated by an interest in investigating certain intriguing questions in the genesis of the general theory of relativity.

[6]Rest and *acceleration* with respect to an *inertial frame* is, of course, tacitly assumed.

[7]This bold intuitive extrapolation is of course a remarkable characteristic of Einstein's youthful genius.

[8]Rather than "in the two directions perpendicular to this direction" [in the original: . . . in den beinden dazu senkrechten Richtungen . . .] what was intended is, of course, "in any direction perpendicular to this direction" (reflecting the cylindrical symmetry about the direction of acceleration).

[9]What is to be taken as "small" for the dimensional quantity γ is apparent from subsequent discussion.

[10]The original word "nutzbar" is replaced in the second reference of Ref. 1 by the word "ersetzbar."

[11]See Ref. 1 or Ref. 2.

[12]At the end of the second reference in Ref. 1, Einstein states that he is prompted by a communication from Planck to clarify the notion of "uniform acceleration" in the new kinematics; and that this is to be understood here as the acceleration relative to the instantaneous rest system of the body under consideration. He concludes that the "relation $v = \gamma t$ holds only in the first approximation; but this suffices, since only terms linear with respect to t or τ need to be considered here."

[13]This restriction would be, of course, already contained in the title of the section, if there existed no ambiguity in the use here of the term "acceleration." But, actually, such an ambiguity does exist (see footnote 12). For an explicit relativistic definition of *uniform acceleration*, see, e.g., W. Pauli, *Theory of Relativity* (Pergamon, New York, 1958), Sec. 26; or H. M. Schwartz, *Introduction to Special Relativity* (McGraw-Hill, New York, 1968), Eq. (ii), p. 83.

[14]See Ref. 1 or Ref. 3. As in Ref. 2, but not in Ref. 3, Einstein's notation is retained here throughout. The symbols X, Y, Z and L, M, N represent the respective components of the electric and magnetic field intensities.