Analysis of André-Louis Cholesky's Manuscript "On the Numerical Solution of Systems of Linear Equations"

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As an Artillery Officer and *École polytechnique* student (X1895), André-Louis Cholesky was both confronted with first-hand reality (such as at the military front where he was eventually killed in 1918) and the demanding nature of making his theoretical ambitions official. For this reason, when duty called him to various geodetic works in France and abroad, he took the time to write up a text to explain the method (we will not yet use the term algorithm) that he had just developed for simplifying calculations. This is the manuscript (which dates back to 1910 but remained in family records until 2005) that allows us to rediscover his works[1]. While we are already fairly aware of this method, it is this text (which is in a version that is ready for publication) that will shed some light on the theoretical and practical ambitions of Cholesky rather than just the simple result.

Figure 1. André-Louis Cholesky in 1917

1. These works were known before the manuscript's rediscovery thanks to a note from Commander Benoît, Cholesky's colleague in the geographic services of the army: the note was on a method for solving normal equations by applying the method of least squares to a system of linear equations of a number less than those of unknowns (the process of Commander Cholesky), *Bulletin géodésique*, 2(1924) 67-77.

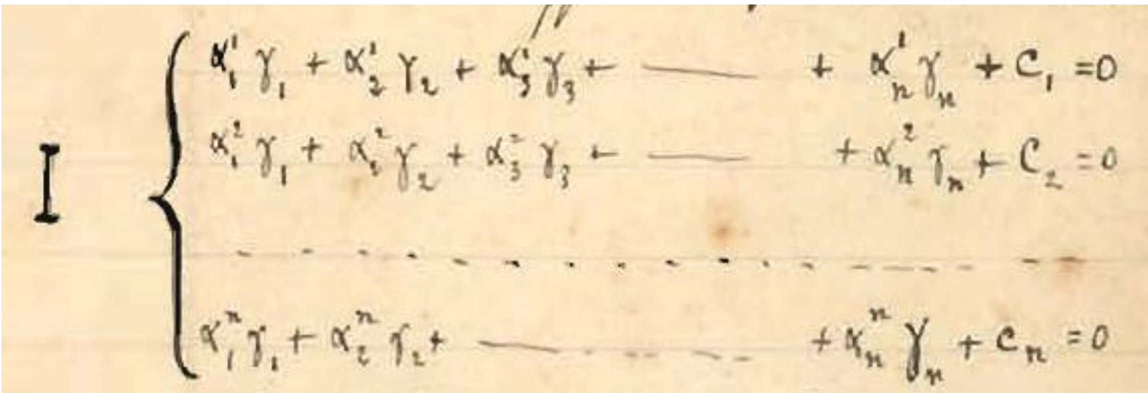
The introductory paragraph reminds us of the necessity of the treatment of "experimental data". In "the case for the adjustment of geodetic networks" (which especially interested Cholesky, who took part in topographic studies in France, North Africa and Romania...) or other "research on the laws of physics", we often return to a numerical solution of the heavy linear systems whose coefficients derive from physical measurements. The preliminary step here is to apply "the method of least squares". If the system accepts a greater number of equations than unknowns (as is the case for geodetic topographic studies), it is possible (and even common if we take into account the measurement errors) that it does not accept exact solutions. Carl Friedrich Gauss developed, in order to refine the position of the asteroid Ceres[2], an effective method for an approximate solution: in order to "solve" at best the system AX=B where A is the matrix of (measured) coefficients of the system, X the vector of unknowns and B a second member vector (often from measurements), it is enough to minimise the function $X \rightarrow ||AX-B||^2$ which, to a vector X, includes the sum of the squares of the AX-B components (the Euclidean norm is then interpreted as a square error). This minimum is then obtained for the vector X solution of the system of so-called normal equations (there is this time as many equations as unknowns) ${}^tAAX={}^tAB$. Note that the initial situation can thus be reduced to a square system of which the matrix $M = {}^tAA$ is symmetric (a point that will be fundamental to the detailed algorithm that follows). Cholesky does not mention it but the then obtained matrices M are also positive (that is to say, that for any column vector X, ${}^tXMX$ is a positive number) and, in practice, the majority are also invertible – what he uses without much clarification is the rule in the geodetic calculations where there happens to be more linearly independent equations than unknowns.

A Matrix and its Transpose

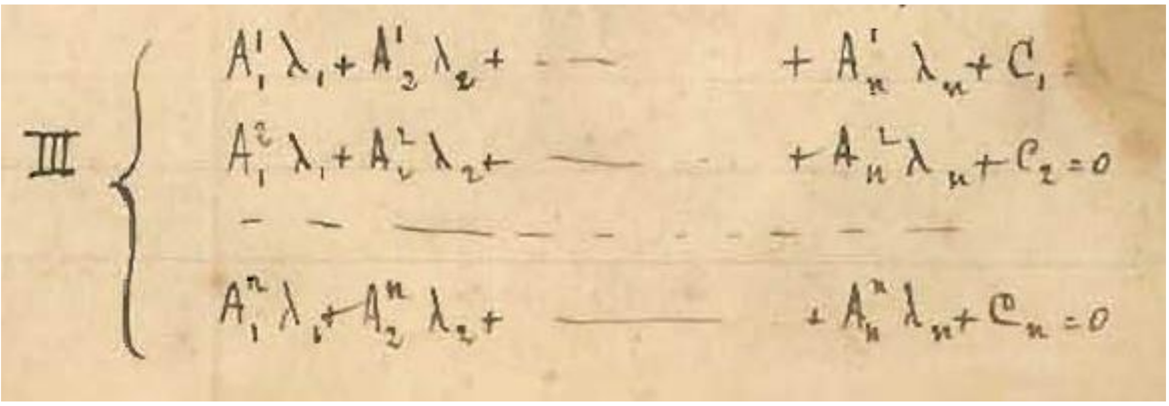
We define for any matrix M a new matrix called the transpose of M, denoted ${}^tM$, by switching the roles of rows and columns: the coefficients of the row of index *i* of M are the coefficients of the column of index *i* of ${}^tM$. Matrices equal to their transpose have many properties: they are "symmetric" since the coefficients in symmetric positions in relation to the main diagonal are equal.

2. In January 1801, Italian astronomer Giuseppe Piazzi discovered in secret an asteroid that he named Ceres. Unfortunately, after 41 nights, he lost sight of it in the glare of the sun. Gauss assumed that Ceres had an elliptical orbit, by observing the parameters via the method of least squares, and was able to predict the time and location of its reappearance (which Heinrich Olbers would observe). Despite his prowess at the observatory in Göttingen, when Gauss began his geodetic work, he came across the same problems with linear systems.

In order to follow the method of Cholesky, we will denote T as the matrix of coefficients $(\alpha^i_j)_{i,j}$, Y the column vector of coordinates $(\gamma_j)_j$, and likewise X of coordinates $(\lambda_i)_i$ and C of coordinates $(C_i)_i$. If we know how to solve a matrix system (the system (I) given in figure 2, which can be rewritten as TY+C = 0),



then by these same operations (as has been indicated), we know how to solve the matrix systems ^tT and thus, in particular, the system (II) $Y={}^tTX$. As a result, by combining these two steps (which are in fact but the one calculation step), we can get the same solution to the system $T^tTX+C=0$ (the system (III)):



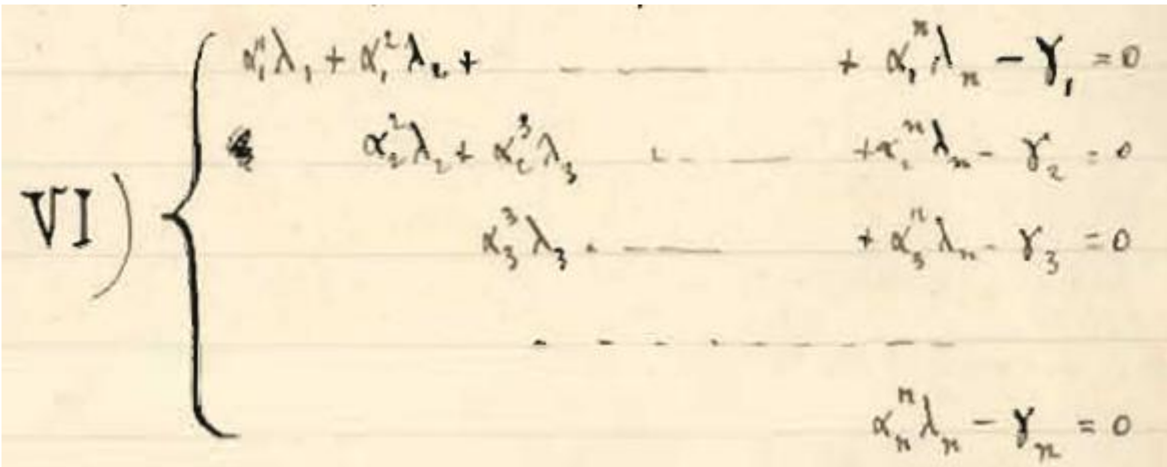
Furthermore, the matrix T^tT, whose coefficients are denoted as A^q_p by Cholesky, has the advantage of being symmetric: "$A^q_p = A^p_q$". These seemingly insignificant comments indeed prove to be at the heart of the method: for if a symmetric matrix A (such as those that occur naturally after the minimalization of least squares) is written as $A=T^tT$, then, consequently, in order to solve a matrix system A, it is actually enough to know how to solve a matrix system T.

Therefore, we propose to solve a system of equations of the form III.

In this objective, set out by Cholesky, the method is now clear: we must decompose the matrix system (III) as a product T^tT where T is a matrix of a system (I) that can be solved easily. Cholesky's idea involves placing himself in "the case where the first equation contains only γ_1, the second γ_1, γ_2, the third γ_1, γ_2, γ_3, and so on", i.e., in the case in which the matrix T is lower triangular.

In order to obtain T from the matrix A, it is enough to write the n(n+1)/2 equations (that are non-linear), obtained by identifying A and T^tT, of which the unknowns are the coefficients of T. Cholesky subsequently noticed that it is actually easier to solve these equations by considering them one after the other:

> *We therefore see that all coefficients of the system VI can be calculated line by line.*



However, he fails to mention that he is actually making a hypothesis: in order to be able to find all coefficients of the line p, we need to be able to divide by α^p_p, and it is therefore important that this coefficient is not zero. This actually comes from the invertible character (or the positive character, as equivalently defined) of the matrix of the system that is obtained by the method of least squares with many independent equations. This concludes the explanatory phase of the method, allowing for the effective solution of symmetric linear systems.

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Yet, Cholesky is not content with just providing a formula; he also strives to explain how to carry out the calculations and comment on his own algorithm as an ahead-of-his-time numerical analyst. For this reason, the following paragraph details how, through a table arrangement of the calculations and appropriate use of a calculating device, we can make the application of the method more reliable.

"Dactyl"-type machines are calculating devices built around a wheel whose number of cogs varies during calculations, invented at the end of the 19th century by a Swedish engineer. The characteristic that justifies the statement of officer Cholesky is the display quality that indicates the sign of the result obtained by a simple readable colour code, among other things.

After this educational precision, Cholesky adopts the role of mathematician:

We can highlight the advantages of this method of solving linear systems in terms of the approximation with which the results are obtained.

More specifically, he goes on to compare the error made in carrying out the calculations at "limited precision" (i.e., with a fixed number of decimals) in his method where $A=T^tT$, and in a method where we would have decomposed A, written as LU with L and U as two triangular matrices of coefficients (β^q_p) and (δ^q_p) retrospectively. By identifying the expressions obtained for the coefficients of A, we thus verify that the squares of the coefficients of T in the first case are equal to the products of the coefficients of L and of U in the second case, which results in the notations of the manuscript: $\beta^q_p\delta^q_p=(\alpha^q_p)^2$. An error η on each of the measurements leads to, in the preceding equality, errors of $(\beta^q_p+\delta^q_p)\ \eta=2\ \alpha^q_p\ \eta$. Yet, the minimum of the function $(x,y) \rightarrow x+y$ for the pairs (x,y) verifying $xy=a^2$ is obtained for $x=y=a$, which Cholesky himself expresses as the following:

We know that the product $\beta^q_p\delta^q_p$, with the sum of its two factors being constant, reaches its minimum when they are equal. Therefore, the slightest error that we could introduce is $2\ \alpha^q_p\ \eta$.

Cholesky can thus deduce that it is his method that is optimal (for the reduction of the error made as a result of imprecisions of measurement) among the methods based on the decomposition into products of triangular matrices[3] :

As a result, the mode of resolution of linear systems that has just been explained appears as the one that provides the best approximation of the calculations.

Subsequently, Cholesky explains how to retain the benefit of this method by approximately carrying out the n square root extractions from a "table of squares" and some practice. The method that is proposed is attributed to Hero of

3. Yet, it should be mentioned that the methods for LU-type decomposition are essential for decomposing non-symmetrical matrices. Actually, all square matrices can be decomposed as PLU, where P is a permutation matrix (and therefore a permutation of the rows), L a lower triangular matrix and U an upper triangular matrix.

Alexandria, a mathematician of the first century AD, but Cholesky seems to ignore it. He explains how to obtain the root of a number N by recognising an approximation n of error ε. As a matter of fact, in the first order the relation $N=(n+\varepsilon)2$ becomes $N=n(n+2\varepsilon)$. From there, we can deduce an expression of ε, and then $n+\varepsilon=(N/n+n)/2$: the approximation n has been replaced by a more precise approximation $(N/n+n)/2$. Therefore, by repeating this process (which, in more mathematical terms, is the calculation of the recurrent sequence that is defined by the function x $(N/x+x)/2$), an approximation is obtained with the required precision, provided that the first approximation is sufficiently reliable.

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Cholesky, still concerned about the practical application, proposes a step-by-step method of verification for solving systems, which is far from being unnecessary when the solution time comes close to “4 or 5 hours”: it is a question of marking at the end of a row, opposite the sum of coefficients of the corresponding row. By carrying out the operations on the rows as well as in the last column, we must verify for each stage that the sum on a row (the size of a coefficient) is zero: this invariant of calculations therefore allows us to eliminate any resolution errors, although it does nothing for the Cholesky method itself.

Finally, Cholesky recalls the concrete application of his method in his geodetic calculations:

> *By this method, several systems over 30 equations were solved and in particular, a system of 56 equations. This last case is part of an adjustment calculation of the altitudes of the primordial chains of the triangulation of Algeria.*

This exact method of solution is even more admirable that it contrasts with the repetitive methods developed in the nineteenth century by Gauss, Seidel and Jacobi, even though it also relies on an earlier minimisation in the sense of least squares. Due to the quality of this numerical analysis text, and the attention paid to pedagogical explanations, this text could be considered an avant-garde model.

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(September 2008)

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