Villarceau Circles

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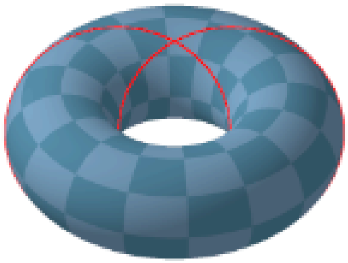


***Figure 1:* Villarceau Circles on a torus.**

For me, it is a great pleasure to write a text about Villarceau Circles[1], because they are doubly and strongly linked to my personal life. Here is why. My first encounter with these exotic circles came about like this. In my final year of high school in 1943-1944, in addition to the typically recommended maths books, I had at hand the work that my father had had in the very same class, the exact title of these two great volumes being *Traité de Géométrie* by authors Rouché and de Comberousse. Certainly, they fully treated the geometry programme of final year science, but they were also filled with out of programme appendices (at that time important because they were linked to posterior geometry). Slightly obsessed with geometry, leafing through the second volume, I discovered in an appendix this surprising fact (an understatement), at least for me but probably for you too, dear reader, that a torus (always understood at the time as a torus of revolution), contained many other circles than meridian and parallel circles. I was so surprised that, instead of watching the demonstration, and in great laziness, my first reaction was to try a physical demonstration, namely to saw a ring from a wooden curtain rod. With primitive handsaws, and mainly with a ring that was ultimately too thin, the operation proved inconclusive. Regarding an elementary demonstration, and others at all levels, whose data of this result in preparatory class still remains spectacular, see

1. Villarceau (whose full name was Antoine-Joseph Yvon Villarceau) was actually an astronomer (see biography). A street in Paris bears his name (XVI° arrondissement).

below. Anyway, then I had completely forgotten about these circles and buried myself in a thesis and other differential geometry works (of all sizes).

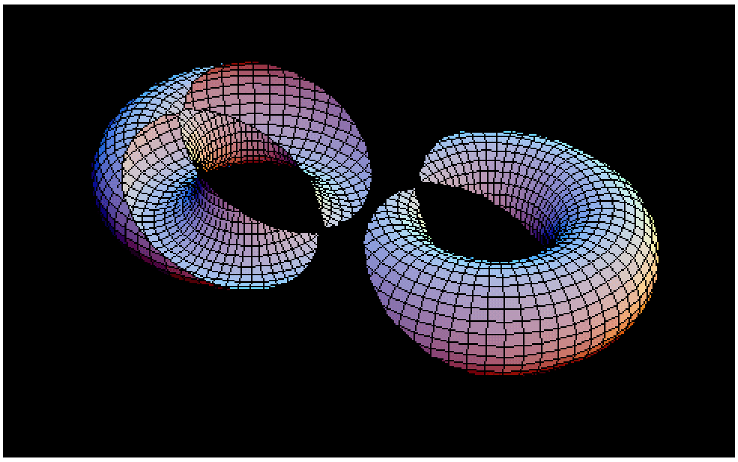
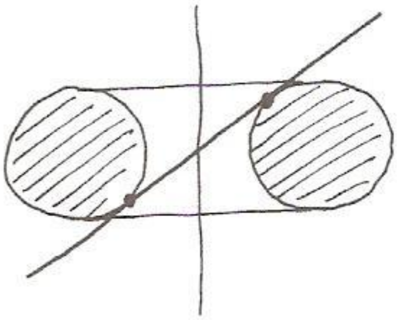


***Figure 2:* Torus cut by a bitangent plane (tangent above and below).** *Here is the cut that the author had failed to make, armed with his saw, on a curtain rod ring. We separate the torus into two pieces, and on the left section we see Villarceau circles appear, with their two bitangent meeting points (top and bottom) (image Académie de Nancy-Metz).*

***Figure 2a:* Representation of the bitangent plane, vertical cut from the operation carried out on the torus** *(drawing Marcel Berger)*



My second encounter came about like this. Equipped with this theory and a lecturer position in Strasbourg, where I visited the Musée de l'Œuvre Notre-Dame, which is essentially devoted to medieval sculptures. I accessed the first floor by a beautiful spiral staircase, when I discovered a torus-shaped carved stone, right at the top of the stairs. In the photo below the sculpture's edges are clearly seen, but not believing it with my eyes I checked that these edges are actually in planes, and are therefore Villarceau circles. Lacking courage, and without knowing how to verify if this 'Villarceau' sculpture was just a fluke or if the sculptures did indeed know these circles, I have since abandoned the idea of finding out the answer. But regarding these circles, and the geometric questions and results that they have subsequently caused, I have never failed to include this photo, and a fairly long text to accompany it, in all my geometry books where Villarceau quite naturally crops up [2].

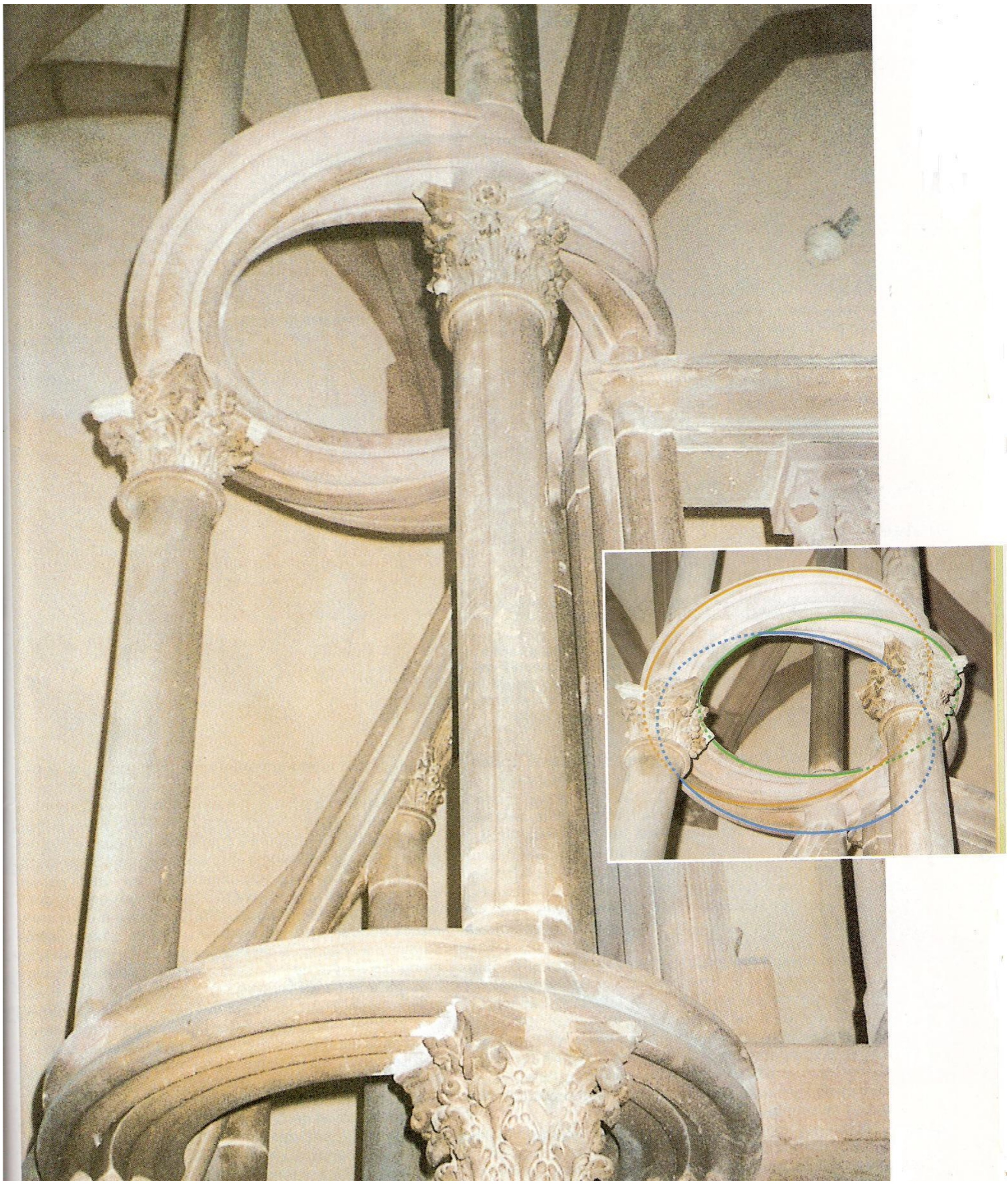


***Figure 3:* Stairs at the Musée de l'Œuvre Notre-Dame, in Strasbourg.** *The first level (bottom) consists of concentric circles (parallel to the torus). The second level (top) is more interesting. First of all, like the other, it is horizontal, contrary to what a perspective effect of this photo, taken from below, may suggest (we are convinced by it considering that the columns connecting both levels are necessarily the same size). Then, it is composed of Villarceau circles: on the right, inset, the three Villarceau circles represented by the stairs' architect, Thomas Uhlberger - these are non-intersecting circles, entwined, belonging to the same family. Contrary to what the photograph leaves to appear, these circles are really quite small - the diameter of tori rarely exceeds 70 cm (photo Michel Pfeiffer,* Pour la Science, *n° 292, February 2002)*

2. For example, see Berger, *Géométrie*, section 10.12, or Berger, *L'échelle de Jacob*, p. 121.

More importantly, Villarceau circles are obtained by cutting a torus (of revolution, with the exception of some, i.e., the two radii that determine it, to an isometry of any nearby space) by drawn plans of cross-section in figure 2. This flat section of the torus is composed of two circles, instead of just being a curved plane, of degree four like the torus itself. This separation into two circles, while certainly intersecting, is the unexpected phenomenon. The revolution around the axis therefore produces two families in a parameter of circles situated on the torus, in an "oblique" sort of way. Presupposing that the regular reader is as lazy as me, but as full of easy curiosity, first of all, here are main properties of these circles, which are fairly unexpected. We will return to the varied evidences of this phenomenon later – although to understand that they are two different families, continuous and at a parameter of these circles (fans of quadrics will be able to remember two families of hyperbolic paraboloid and hyperboloid generators).

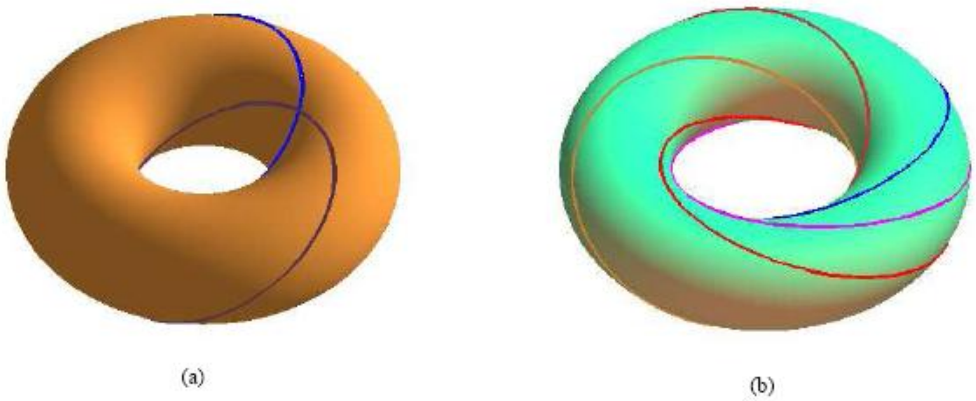
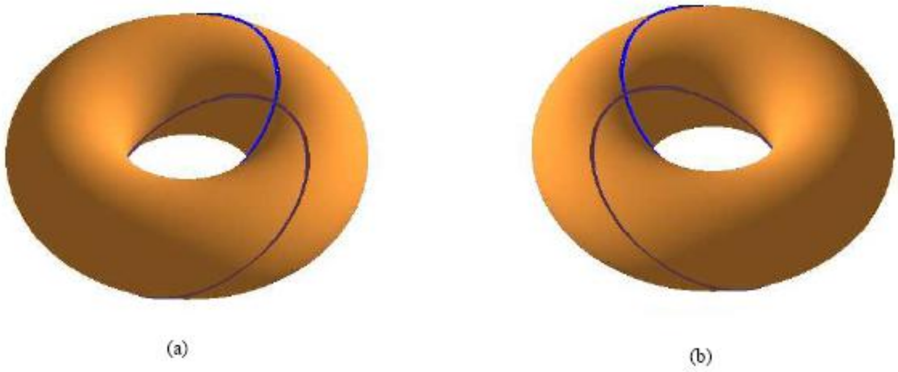


***Figure 4:* One of the two families of Villarceau circles (on the right).** *Figure (a) on the left represents the two Villarceau circles from a bitangent plane cut, defining both families. The rotation around the torus of the upper light blue circle in (a), or the rotation of the bitangent plane, gives the family of circles shown in (b), on the left. It is a question of one of the two families of Villarceau circles. Note that the lower dark blue circle (a) does not coincide with any of the circles in (b): it belongs to the second family. The two families are different because, unlike meridian circles, there is no invariance (return to the same position) at 180°: the return to the same position is done by complete rotation at 360° and defines two families of circles.*

***Figure 4a (below):* No invariance by rotation of the bitangent plane at 180°** *(the position is not the same).*

(images Lionel Garnier, University of Burgundy, Revue électronique francophone d'informatique graphique*, 2008)*



THIS ARTICLE'S STYLE

Our desire is to please the reader: that they may find certain geometric joy, and no longer get lost among all the demonstrations unless this really pleases them. In other words, we prefer to state geometric properties without demonstration, quite visible most of the time, with the notable exception of the "modern" geometry dispute.

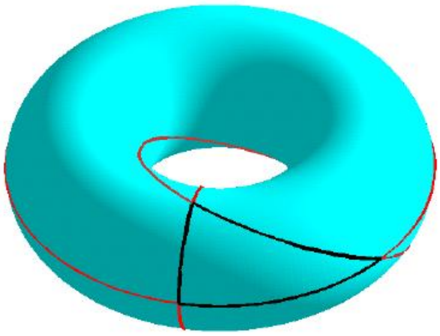


Figure 5: ***3D triangle rectangle with circular edges, on a torus, whose edges are a meridian arc, a parallel arc and a Villarceau circular arc*** *(image Lionel Garnier, University of Burgundy,* Revue électronique francophone d'informatique graphique*, 2008)*

The first property of Villarceau circles is enlacement. First of all, unless explicitly stated otherwise, notice that in all that follows the considered torus is ordinary. So, we have two distinct circles of opposite families that always meet each other at two distinct points. On the other hand, the behaviour of two distant circles of the same family is much more subtle. Not only do they not meet, but they are mainly always *enlaced*. Certainly, it is obvious on the figures for the non-meeting, demonstrated in fact in a simple way (i.e., left to the reader), but the question of enlacement is another kind of difficulty. We leave the reader to mathematically define two simple closed curves of the *enlaced* space. This tale is quite interesting - to our knowledge Gauss was the first who took care of these questions on enlacement, as well as the knot theory while we are at it. Typical of mathematics, the idea is to connect any pair of curves (always simple closed) with an *invariant,* known as many enlacements of these two curves, invariant to what we ask to be non-existent if the two curves are separable - if it is not non-existent, then the two curves are not separable, they are interlaced [3].

The second property of Villarceau circles is a loxodrome. A property, common to all these circles is that every Villarceau circle is a *loxodrome* from the torus. Remember, if necessary, that a loxodrome on Earth, on a surface of revolution, in a more mathematical sense is a curve that cuts all meridians (and therefore all parallels too) at a constant angle. This notion remains today, with inertial measurement units for naval or air navigation, but in particular with GPS. But before these means of navigation, both at sea and in the air, loxodromes were fundamental. Indeed, by definition in some way, when we navigate "in a constant direction", we make sure that the compass always marks the same direction, i.e., the same angle as the magnetic meridian[4]. Note that the loxodromes of a torus are only Villarceau circles for this ("one") value.

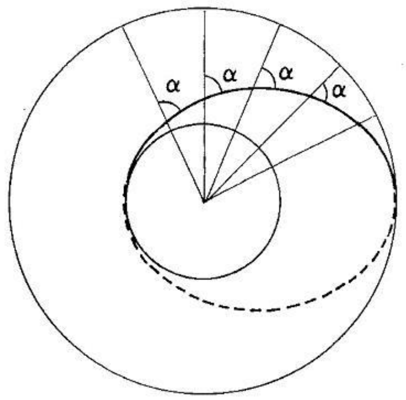


Figure 6: **Loxodrome property.** *The Villarceau circle (which appears as an oval in this perspective view from above) cuts each meridian with the same angle, based solely on characteristics R and r of the torus (figure Marcel Berger,* Géométrie, tome 2, *Cédic Fernand Nathan, 1979).*

The complete history of Villarceau circles is not well known. I have no reference: it seems that, for example, this loxodrome property has been known since 1881. But who was the first to discover this loxodrome property? I would

3. We will find in Berger-Gostiaux, sections 7.4, the definition of this number. The interested reader will be able to calculate this invariant for two Villarceau circles, if not only to prove that it is non-existent.
4. We know that the orthodrome (to follow a great circle of Earth, by boat or plane) is the shortest navigation, but it obliges to permanently change direction. The loxodrome is an easier navigation (we keep the same direction), but this route is longer.

appreciate other references. Also, I have no idea of how Villarceau discovered the eponymous circles. His 1848 note, less than a half page, in the Compte-rendus de l'Académie des sciences is only a curt and quick calculation. The article of the same year in the *Nouvelles annales de mathématiques* said nothing else, except the expansion to the case of surfaces of revolution created by any conic section – writing just as devastating.

Properties of Villarceau Circles: Better Than the Loxodrome, the Parataxis

We can reread this loxodromic property as such: all meridian planes, i.e., the planes passing through the axis of the torus, cut a Villarceau circle at the same angle. We say that Villarceau circles are *helices* of the torus. Again, we do not know who first discovered this property and when. It belongs to the domain known as the *parataxis.*

This property generalises any real number α the trivial fact that any plane containing the axis of a circle cuts this circle at a right angle. In addition, any sphere containing a circle always cuts the axis of the circle at a right angle. It is the particular figure of two circles forming an *orthogonal ring*. Each sphere containing one cuts the other at right angle. More generally, a pair of circles is found to be paratactic when any sphere containing one cuts the other at a constant angle, and then reciprocally, – *i.e.,* this condition is symmetrical: any sphere passing by the second circle again cuts the first at a constant common. The result here is that any pair of Villarceau circles of the same family is paratactic (for an angle depending on the pair).

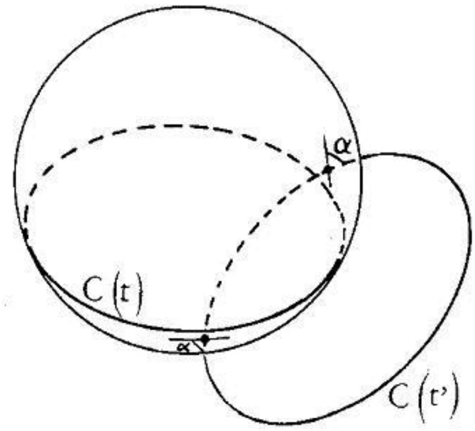


***Figure 7:* Parataxis property.** *Any sphere containing a Villarceau circle cuts another Villarceau circle of the same family at the same angle α the loxodrome (figure Marcel Berger,* Géométrie, tome 2, *Cédic Fernand Nathan, 1979).*

A brief word on the parataxis: it is found exposed with the correct conceptual language in the work of Felix Klein and the German geometry school from 1900[5]. *A contrario*, the "classic" French geometry school, losing speed between 1850 and 1950, maintained, particularly for the parataxis, a teaching and didactic works of a fairly ancient "style"; for example, see Hadamard, Dontot[6], etc. I had the opportunity to narrate this French mathematical "gap"[7]…

And Demonstrations? The Naïve but Explicit

Villarceau's original demonstration was analytical; it is extremely easy if you know the result, because we roughly find the radius of the centre of these circles, thus their equation in the flat section, and we verify that this section's 4th degree equation is divided well into the product of these two circles' equations.

The analytical evidence is developed in the box below. Then, in the next section, we will give evidence that has had considerable success, even today, to introduce several new concepts, namely the complexity of a space or a regular plane, in addition to points at infinity[8].

Cartesian and Parametric Equations of the Torus and its Villarceau Circles

Nevertheless, we give some equations that Cartesian readers would like (starting with coordinates, rightly known as *toric*).

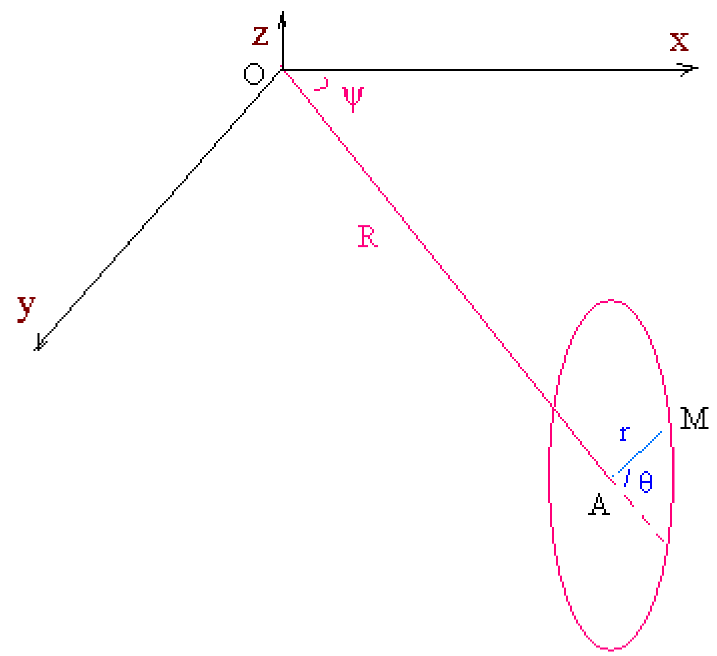
A point M of the torus is parameterised by the angles ψ and θ as follows:

$$x = (R + r\cos\theta)\cos\psi$$
$$y = (R + r\cos\theta)\sin\psi$$
$$z = r\sin\theta$$

In order to remain in pure Cartesian coordinates, we project M on axis OA at a point m (not shown) of coordinates (x,y,0). The torus equation is written $AM^2 = r^2$, either $(Om - OA)^2 + mM^2 = r^2$, or:

$$(\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2$$

5. This paper can be found, i.e. in the correct language, particularly algebraic, in Berger's sections 10.12, 18.9 and 20.7, *Géométrie*. We do not really understand what is happening without three essential concepts for the geometry of the circles and spheres of space: the space of all spheres, the pentaspherical coordinates, the conformal group of the sphere, and then comes the idea of going through stereographic projection of ordinary space to the three-dimensional sphere, which has the advantage of being compact.
6. Jacques Hadamard, *Leçons de Géométrie élémentaire*, Armand Colin 1932 ; René Dontot, *Étude élémentaire de la parataxie et des cyclides*, Vuibert 1945
7. Marcel Berger, *Cinq siècles de mathématiques en France*, ADPF (Association pour la Diffusion de la pensée française), 2005.
8. An informal approach of the Villarceau circles can also be found, by rotation around the axis of the cutting planes of the torus, from the vertical plane to the vertical plane passing through the plane of bitangence, in Alexandre Moatti's blog, "Les cercles du tore".



We find this equation by using the toric coordinates above. The most general equation, taking into account the other part of the torus, symmetrical, in relation to O, is written:

$$(\sqrt{x^2+y^2}\pm R)^2+z^2=r^2$$

We eliminate the radicals as follows:

$$x^2+y^2+z^2-r^2+R^2=\pm 2R\sqrt{x^2+y^2}$$

hence the Cartesian equation of the torus, of fourth degree:

$$(x^2 + y^2 + z^2 + R^2 - r^2)^2 = 4R^2 (x^2 + y^2)$$

We can get back to the two equations above with the aforementioned torus parameterisation ψ and θ.

@@@@@@@

Now to discuss the research of Villarceau circles, even if it is easier when we know that they exist and that we know their respective centres and their radius!

For the sake of simplicity, the tangency in M_T in a plane (x,z) is expressed by an angle α between the bitangent plane and the main parallel plane of the torus such as (see figure 2a for the trace of the bitangent plane): $\sin\alpha = r/R$. In the plane (x,z), the point M_T of contact has for coordinates ($OM\cos\alpha$, $OM\sin\alpha$); therefore, it verifies the equation: $x\sin\alpha = z\cos\alpha$, or $rx=\pm R\sqrt{1-r^2/R^2}\, z$ (the sign $\pm$ recalls that in a given cut (x, z), there are two bitangent planes, symmetrical in relation to the x axis. Consequently, the bitangence is expressed by $r^2x^2 = (R^2 - r^2)\, z^2$.

The first Villarceau circle regarding this configuration has for its centre the point (0, r, 0), the other (0, -r, 0). Note that it is a circle, of radius R, i.e., verifying the equation $x^2 + (y-r)^2 + z^2 = R^2$ (in fact the Villarceau circle is going to be the intersection of a sphere with the bitangent plane). For this, parameterise it as follows $y = r + Rt$.
The equation of the sphere above gives $x^2 + z^2 = R^2 (1 - t^2)$
When we associate it with the cut by the bitangent plane, we obtain

$$x^2[1 + r^2/(R^2 - r^2)]= R^2 (1-t^2)$$

$$x^2 = (R^2 - r^2) (1-t^2)$$

Yet, the characteristic equation of the torus is:

$$f(x) = (x^2 + y^2 + z^2 + R^2 - r^2)^2 - 4R^2 (x^2 + y^2)$$

$$f(x) = [(x^2 + z^2) + (r+ Rt)^2 + R^2 - r^2)]^2 - 4R^2 [x^2 + (r + Rt)^2]$$

$$f(x) = [R^2(1-t^2) + r^2 + R^2t^2 + 2Rrt + R^2 - r^2]^2 - 4R^2 [x^2 + (r +Rt)^2]$$

$$f(x) = 4R^2 [(R+rt)^2 - x^2 - (r +Rt)^2]$$

Returning to the value of x for the bitangent plane above:

$$f(x) = 4R^2 [(R+rt)^2 - (R^2 - r^2) (1-t^2) - (r +Rt)^2]$$

Ten term equation from which it is verified that they cancel each other in pairs. The point in question really belongs to the torus.
The intersection of the "Villarceau sphere" of centre (0,r,0) and of radius R with the bitangent plane gives a figure whose every point (parameterised by t) belongs to the torus. The intersection of a sphere and an intersecting plane being a circle, all these points therefore represent a circle on the torus: the Villarceau circle.

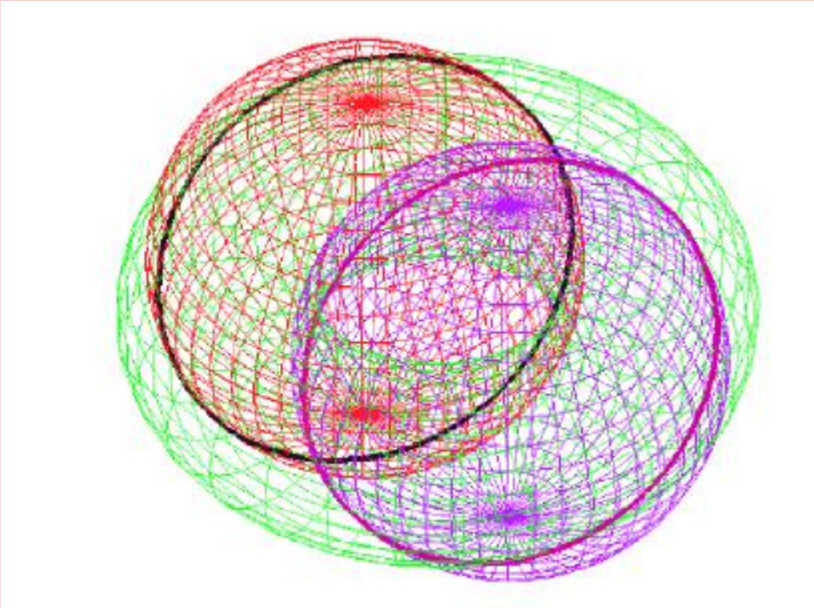


***Figure 8:* The Villarceau circles can also be seen as the intersection of two "Villarceau spheres" with the torus.** *These spheres are symmetrical in relation to the vertical axis of the torus, their centre is at a distance* r *(small radius of the torus) of the central point, on both sides; their radius is* R *(large radius of the torus) (image Lionel Garnier, B. Belbis, S. Foufou, University of Burgundy*, Revue électronique francophone d'informatique graphique, *n°3, 2009)*

THE DEMONSTRATION OF "PURE" AND REALLY ILLUMINATING GEOMETRY

This is for me quite a unique opportunity to explain the term *modern geometry*, both its pros and cons. Here is how things were presented when I was in preparatory classes in the mid-1950s. And nobody found fault with it, at least in the scientific circles, which I will discuss again. Here is the *verbatim* (patience if you do not understand even simply what is at stake).

First of all, we must see that in the Euclidean plane, when we duplicate it using two complex numbers as coordinates, and add to it points at infinity, in this space (four-dimensional) the extension of a circle becomes a curve that always contains two points at infinity called "circular points". This is because the equation of a circle always starts by $x^2 + y^2 + \ldots$, and so, in this **double generalised** space, two circles in generic position always have four points in common: in coordinates (x,y) extended in (x,y,z), but only to consider a nearby scalar, for having points at infinity (x,y,0), our cyclical points are written (1, $\pm$i ,0), where *i* is the classical complex number ($i^2 = -1$).

A 3-dimensional space that is both complex and has points at infinity is now placed in the domain of geometry (therefore it is six-dimensional). Our torus is algebraic (i.e., an object defined by a polynomial) and of degree 4, therefore any flat section will also be algebraic and of degree 4. But the torus contains, in two ways, the umbilical, namely the points at infinity defined by $x^2 + y^2 + z^2 = 0$ (see below, figure 9), because we have a surface of revolution and with circular merdians. The section of the torus by our candidate Villarceau plane contains four double points, two for the points of contact (they are neither at infinity nor complex) and two others that are the points of intersection of this double umbilical curve with the considered plane. Yet, algebraic geometry tells us that a four degree curve, with four double points, is necessarily degenerate into a pair of conic sections, evidently of circles. But this explanation is as clear as mud[9]!

9. Evidence of "modern geometry" is found in Lebesgue, both faster but conversely harder to completely implement for passages to go to and from the complex. Namely an inversion is made. One of the cyclical points of the meridian circles is then taken as the centre of inversion. Then the opposite of the torus becomes a cone of revolution, and the plane of section becomes a bitangent sphere to this cone. Their intersection is then elementarily seen as splitting up into two circles.

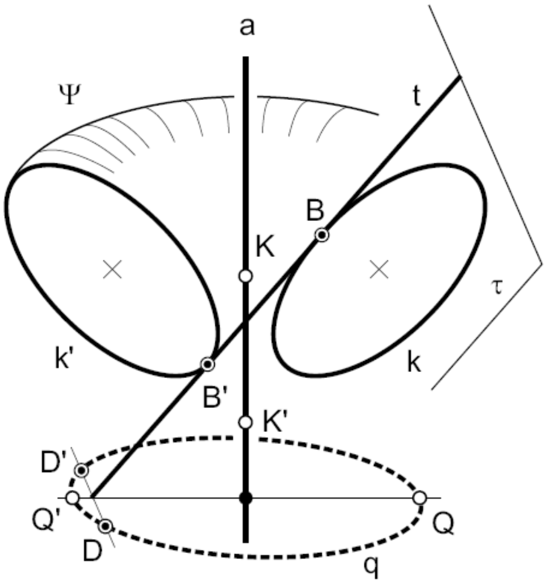


***Figure 9:* Demonstration of projective geometry.** *In this figure, in part doubly symbolic (because there are imaginary elements and elements at infinity), ψ designates the torus, a is the torus axis, k and k' designate the two meridian circles of the frontal plane, B and B' designate the two points of contact with the torus of the considered bitangent plane, noted τ, t is the straight-line of the plane τ (like in figure 2a). All are real but are now, like the meridians, pairs of circles (k and k'). Their cyclical points Q, Q' (imaginary and at infinity) create a curve, counted twice, noted q (always at infinity and imaginary, known as umbilical) and drawn in bold lines to help this impossible vision. The considered plane τ cuts it in two points D, D'. These points are double points for our section, because the umbilical is described two times.*

(figure Anton Hirsch, Extension of the « Villarceau-Section » to Surfaces of Revolution with a Generating Conic, Journal for Geometry and Graphics *Volume 6 (2002), No. 2).*

Much of Poncelet's work, from 1815 proceeds similarly. Poncelet was a visionary, who worked entirely at ease in complex projective space, and without any use of coordinates. But he gave it no definition, no construction. So, it is not surprising that the Académie des sciences then refused these works, considered "four-dimensional" or "romantic". Actually, the torus of space was considered far worse: it was six-dimensional!

A correct definition of what is complex projective space had to wait until the second half-century, after 1850. Namely what we have written above is all quadruplets (x,y,z,t) of complex numbers, but *only defined to a close multiple*. Points at infinity are those of the form (x,y,z,0). In the demonstration above we obviously forget to say that *in fine* it is in real space, and not projective. So, when we read the texts of these times, we never know if we are in the real, complex, or projective!

The demonstration above was called, in France until very recently, around 1950, "modern geometry". France has seen a huge mathematical delay, particularly between the two wars, in all domains - apart from a some rare exceptions that were practically not mentioned -, while things were done properly in Germany, perhaps surrounding Klein, from 1900. I took delight, during the summer of 1944, from Duporcq's book[10] (of course, with the pain and joy of the exercises), that was recommended to all preparatory class students who wanted to do slightly more than the programme, and even in the final year for talented "addicts" of geometry without coordinates. It is only later that I broke the news, in 1956, to Jean-Louis Koszul of whom I had become the colleague in Strasbourg. And hearing "but, Berger, nothing is really shown in this book!" Well, let's say I lost a certain virginity…

Indeed it would be firmly demonstrated, which is easy to do for this little book, but is particularly made necessary to correct certain "results" of the Italian school of algebraic geometry from the start of the 20th century, obtained in a "modern" way, some of which are proved wrong! It is all "Algebraic geometry" (i.e., of objects defined exclusively by polynomial equations), that it must have therefore been set up from the end of the 19th century…

In fine Two Questions: Are these Natural Questions?

What follows seems to me an excellent illustration of the way in which the great mathematicians function. These questions revolve around the fact that a torus includes four families (always implied to a continuous parameter) of circles. First question: can we find surfaces with more than four families of circles? Second question: what are the surfaces with several families of circles, and how do we characterise them? We are indebted to Darboux from 1880 and his very good text on the question. But it is difficult to read, because we are right in the vagueness of "modern" geometry; it is necessary to hold on to the reading of Darboux in order to know if we are in the complex or the real. Darboux's idea, which works, is that at any point of our surface, the existence of circles contained in the surface and passing through this point implies strong restrictions

10. Ernest Duporcq, *Premiers principes de géométrie moderne l'usage des élèves de Mathématiques Spéciales et des candidats à la Licence et à l'Agrégation*, Gauthiers Villars 1912 (réédition Jacques Gabay 1995).

on the second fundamental form[11] of the considered surface. So, ten families of circles is a maximum - except for spheres evidently with an infinite number of such families. Surfaces with ten families are also completely known.

Surfaces depending only on five parameters are found. It is easy to see that we cannot have more than six such families of circles in the real - but it is ten in the complex. To our knowledge, it does not exist of IT production of these surfaces, when the five parameters are all different, to six families of circles, which result, in addition to Villarceau circles, from the duplication of parallels and meridians[12]. For the active and focused reader on IT devices producing geometric figures, etc., I am interested in figures for the surfaces of six real families.

In any case, it seems to us that this memory of Darboux leaves many questions unanswered, such as how to classify, if at all possible, surfaces with families of less than ten circles...

Digression on Cyclides, Dupin Cyclides and Others

When two of the five parameters are equal, these cyclides are called Dupin cyclides. A characterisation is that these are the only surfaces whose two focal nappe surfaces are actually both reduced to curves, thus a double degeneration.

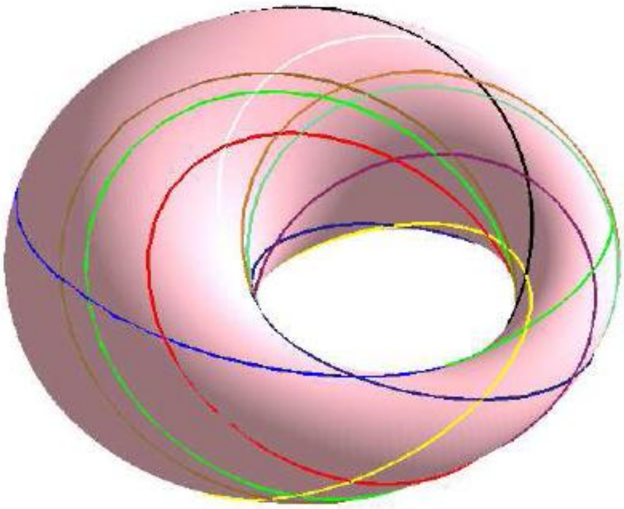


Figure 10: ***Dupin cyclide.*** *Described in 1820 by the French mathematician and politician Charles Dupin, Dupin cyclides are the images of a torus of revolution by inversion (the inversion of centre O and of power k associated with a point M a point M' such as* ***OM'*** *= k* ***OM****/OM², the notation in bold being taken for the vectors). The most*

11. The second fundamental form of a surface (in a point) is the quadratic form that gives, for planes passing through perpendicular angle in this point, the value of the curvature - as curved plane - of the section plane of the surface by such a plane. This form describes how the surface is situated in the space. Darboux shows that the existence of many sections which are circles completely determines the se`cond fundamental form.
12. See their equation in Berger, *Géométrie*, Sections 18.10 and 20.7.2, and in Berger-Gostiaux, sections 10.6.8.2.4 and 10.2.3.

common Dupin clycide is the one above, forming a crescent that is closed. We have illustrated the Villarceau circles, images of those of the torus (the image of a circle by an inversion is a circle) (images Lionel Garnier, University of Burgundy, Revue électronique francophone d'informatique graphique*, 2008)*

Focal Nappes

Reminder of some notions: locally, perpendicular angles at a surface envelope two surfaces, known as focal nappes. The two points of contact on a perpendicular angle are the two main curvatures at the considered point. The main curvatures are a curved line, namely integral curves of the two main directions, which are in turn specific directions of the second fundamental form, and the curvature centres are at distances 1/r and 1/r' on the perpendicular angle, where r and r' are eigenvalues of the second fundamental form. Be careful, see the figure, that these eigenvalues, certainly existant, have a sign once the surface is positioned; in the "convex" case, they are the same sign - but they can also be opposite signs, in the case of a mountain pass, or typically for a hyperbolic hyperboloid.

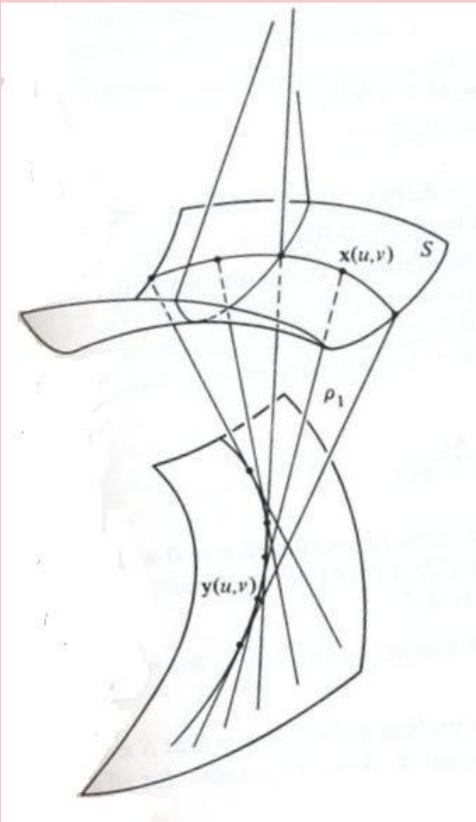


***Figure 10:* Construction of a focal nappe following the perpendicular angles to a surface S.** *When we consider all perpendicular angles to a surface (the perpendicular angle in a point is the perpendicular to the tangent plane at this point), this set can be seen as the set of light rays of an optical device. Apart from the spheres whose perpendicular angles converge at a point, any optical is always imperfect. The focal nappes are*

there where the light rays are gathered (where it burns!). The two points of concentration of light, on a perpendicular angle, generally describe two surfaces, whose points of contact are different, which define the astigmatism of this optical (unfortunately, our eyes, most of the time). In order to find these focal surfaces, it is fundamental to note that, when we consider the perpendicular angle to the surface along any curve, we obtain a ruled surface. But if the curve is a line of curvature, then these perpendicular angles become tangents to a common curve. And reciprocally.

If our two focal nappes are now both degenerated into a curve, and not a surface, then the perpendicular angles along a line of curvature will pass through a fixed point, and the points corresponding to the surface will form a circle, and there will also be a sphere tangent to the surface all along the aforementioned circle. Therefore, our surfaces twice surround the sphere; for the torus, these are the tangent spheres all along a meridian and a parallel. On a nearby conformal map (of products of inversion) we will find our tori [13], along with their Villarceau circles…

■

(March 2010)

(translated in English by John Moran, published July 2013)

The author and the BibNum website thank the University of Burgund's Lionel Garnier for authorising them to use some of his images.

13. Indeed, we easily show when the two focal curves necessarily form a pair of focal conics, i.e., each between them is the place of points such as the cone defined by this point and by basing on the other conic is a cone of revolution. In the case of the torus, these two focal conics degenerate into a circle and its axis (look at the perpendicular angles to a torus!).