Galois, student at Louis-le-Grand school, and his first article

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In 1829, Galois (1811-1832), who was only a young student at the time, published his first article in the Annales de Gergonne which was devoted to equations, and more precisely to "continued fractions". His article attests that it is the work of a student who read and who understood the works of his predecessors who were Eulert and Lagrange in this case. Motivated by this passion for mathematics and encouraged by his teacher – Paul-Émile Richard (1795-1849) – the young man contacted Gergonne to publish his text. In this article, Galois's contribution is analysed and situated in its mathematical and editorial context[1].

ANALYSE ALGÉBRIQUE.

Démonstration d'un théorème sur les fractions continues périodiques ;

Par M. Evariste GALOIS, élève au Collége de Louis-le-Grand.

Figure 1: ***The heading of Galois's article*** *(BibNUm text, digitised by Numdam). Ot indicates : ALGEBRAIC ANALYSIS - Demonstration of a theorem on periodic continued fractions; By M. Evariste GALOIS, student at Louis-le-Grand school.*

1. In order to situate Galois's first text and insert it within the context of his work, we refer to Norbert Verdier, *Galois, le mathématicien maudit*, Published. Belin, 2011.

ENCOURAGED BY HIS TEACHER, GALOIS PUBLISHED HIS FIRST ARTICLE

Having not prepared in the year leading up to his exam, Galois was left disheartened after his first failure to gain admittance to the École polytechnique. However, in 1828-1829, he got into Louis-Paul-Émile Richard's "Special Mathematics" class (see box). Richard started teaching "Special Mathematics" in 1827 and continued teaching it right up until 1848. At that time in Paris, "special classes" had about a hundred students; in 1837, Richard had 94 in his class[2]. There is not any exact information available on the nature of these classes. However we do know that in the face of such a large class, the lessons were mainly taught as lectures and took place every morning between eight and ten o clock (except on Thursdays and Sundays) in accordance with university regulations. Due to these circumstances, there was not any other option but to cut down the preparatory classes for the competitive oral entrance examinations. However, in Paris especially, a certain number of private institutions grew which enabled students to gain oral practice prior to the exam. For example, records state that, in 1832 (the year of Galois's death), one third of successful applicants to the École polytechnique attended one of the four best known private institutions in Paris (Mayer, Barbet, Laville or Bourdon).

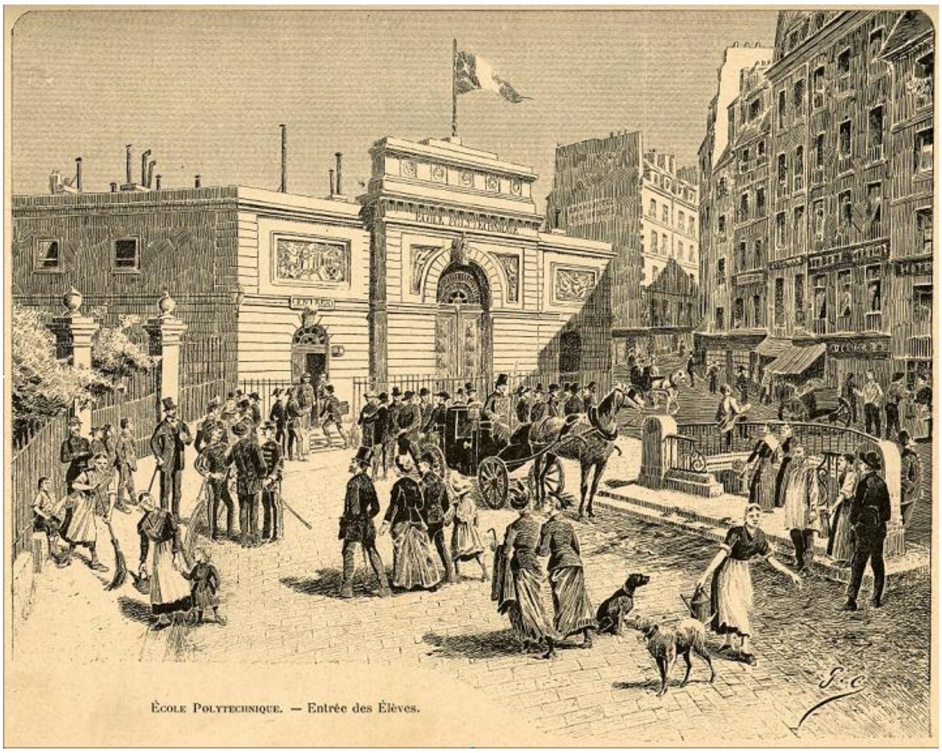

Figure 2: ***L'École polytechnique in Galois's time*** *(image extracted from Gaston Claris,* Notre École polytechnique*, Paris, Librairies imprimeries réunies, 1895). It was*

2. Source from Bruno Belhoste "La préparation aux grandes écoles scientifiques au XIXe siècle: établissements publics et institutions privées" *Histoire de l'éducation*, n° 90, May 2001, 101-130.

founded in 1794 under the name of École centrale des travaux publics and it was the capital of mathematics education in France by 1830. As liberal-minded bourgeois, many students of École polytechnique participated in the events of 1830. *To get into the École polytechnique, one must "be between 16 and 20 years old, have been brought up with religious principles and proclaim them. One must prove their devotion to the King and their good conduct […].One must have had small pox or have been vaccinated […] The annual board and lodging fee is 1000 francs[…] the period of study is at least two years[3]". The students of École polytechnique form a social group that Bruno Belhoste believes to be the basis of what he calls "technocracy", which aims to apply - through schools of application– the theoretical sciences for the benefit of technical and material progress.*

One should not think of these private institutions as rivals of the public sector, at least until the 1850s anyway, but actually as partners. For example, in 1837, three quarters of the students in Richard's class also took classes at Mayer. Galois's poor oral preparation illustrates that he did not attend any of these institutions. However, it should be noted that most of them were developed later. In 1829, Galois re-sat the entrance exam for the Polytechnique but failed again. However, this was not because he had not been noticed. Richard very often spoke highly of him saying things such as: "This student has a clear superiority over all his classmates" and "This student works only with the highest level of mathematics".

Professor Richard's obituary in the *Nouvelles annales de mathématiques* (1849)

"Constantly keeping himself up to date with scientific progress, Richard enriched his classes; his students were eager to work on the questions that he set them; these questions aimed to broaden the mind and not to narrow it which is too often the case. He also trained many distinguished men, several of whom rose to fame. Galois would have been the Niels Henrik Abel of France if a violent death had not brought his short and turbulent life to an end. Mr Le Verrier is known throughout the world for his astronomical calculations. Mr Hermite and Mr Serret, even as only young examiners, ranked among the best French geometricians. Every public service has civil servants of great merit that Richard let into the École polytechnique, and the majority improved their ranking at the entrance exam thanks to a solid preliminary education. Moved by a pure and unselfish zeal for Science, a zeal which is extremely rare, he would support any project which was likely to spread the mathematical truth[4]."

3. Bruno Belhoste, *La formation d'une technocratie. L'École polytechnique et ses élèves. De la Révolution au Second Empire*, Belin, 2003.
4. *Nouvelles Annales de mathématiques*, 1849, 448-452. We refer to the recent study carried out by Roland Brasseur in order to find information on Richard's career: Brasseur, Roland, "Quelques scientifiques ayant

Another bibliographical note, perhaps written by Flaugergues who was one of Galois's classmates, appeared in 1848 in *Le magasin pittoresque* stating "It is not surprising that Mr Richard really valued Galois. The original solutions that this brilliant student gave to the questions set in class were explained to the classmates with just praise for the inventor whom Mr. Richard highly recommended for academic distinction[5]."

It is quite likely that Richard encouraged Galois to publish one of his first works. On 1st April 1829, his "Démonstration d'un théorème sur les fractions continues périodiques" appeared in the *Annales de mathématiques pures et appliquées,* founded by Joseph Diez Gergonne (1771-1859).

THE ANNALES DE GERGONNE AND CONTINUED FRACTIONS

In nineteenth century Europe, mathematics benefitted from a new form of distribution in the form of periodicals which were specifically devoted to the subject. These radically changed communication and exchanges between mathematicians around the globe. The *Annales de mathématiques pures et appliqués* by the mathematician, François Joseph-Diez Gergonne (1771-1859) was published from 1810 onwards and was the first significant journal published on the continent. It was published monthly until 1832 and is now called *Annales de Gergonne*. We will only mention their editor, objectives and origin once[6]. Since the young Galois presented his article on continued fractions in 1829 when he was just a student, we are going to focus rather on the role of school and college students in this journal compared to what was available to them in two other periodicals of the same period.

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Galois is far from being an isolated case and his important contributions to mathematics demonstrate that a student could contribute to the progress of mathematical science via a journal article if they are given adequate

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enseigné en classe préparatoire aux grandes écoles", (season 4), Paul - Émile Richard (1795-1849), *Bulletin de l'union des professeurs de spéciales. Mathématiques et sciences physiques,* 232 (Octobre 2010), 1-8.

5. *Le magasin pittoresque*, 16 (1848), 227-228.

6. Christian Gérini, *Les "Annales" de Gergonne : apport scientifique et épistémologique dans l'histoire des mathématiques*, Published by éditions du Septentrion, Villeneuve d'Ascq, 2002 ; also "Les Annales de mathématiques pures et appliquées de Gergonne", BibNum commented text.

encouragement by their teacher. About 20% of the contributions in the form of articles or of solved problems were in fact the work of this population: college students, secondary school students, students of the École polytechnique, students of the École normale (also called preparatory class at certain periods, particularly during Galois's time) or university students.

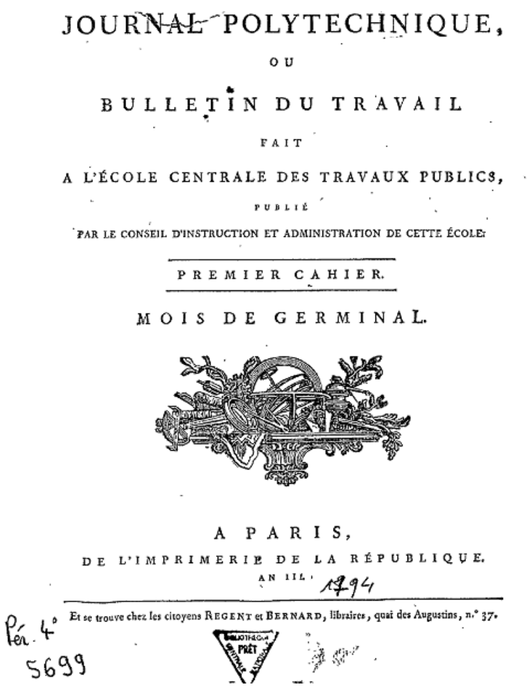
Before issue one of the *Annales* was published in 1810, only two periodicals could offer such an opportunity and only to one category of students; those of the École polytechnique. One of them was *Journal de l'École polytechnique* which was to be published in monthly instalments according to the decree of the 24th Prarial, Year III which stated its objectives[7] and the other was *Correspondance sur l'École polytechnique* by Jean Nicolas Pierre Hachette (1769-1834)[8]. This *Correspondance* was published to fill a gap exposed as early as issue 4 of the *Journal de l'École polytechnique* (1796). It was said that former students lacked a way "to maintain correspondence with the mother school"[9]. The target of a monthly circulation of the *Journal de l'École polytechnique* was far from ever being reached; from 1795 to 1831, only twenty journals were published which is an average of about one every two years. The role of students and former students of the École polytechnique in the journal was negligible as it rapidly became a medium for publishing substantial memoires of the mathematical elite such as Monge, Lagrange and Poisson.

The *Correspondance sur l'École Polytechnique* was also published very irregularly; Issue 1 appeared in April 1804, issue 4 in July 1805 and issue 10 in April 1809. Furthermore, it was largely composed of lists of names such as successful students or those who had been assigned a post within the school. It was also largely made up of letters which did not necessarily have anything to do with the sciences taught at the École, of declarations of official regulatory texts, of lesson plans etc. Therefore, we can safely say that the mathematical articles occupied a relatively small part and were often past lessons of the École which had either been reused or developed. Again, it is easy to see the closed nature of this publication which was eventually reserved for the same elite as the *Journal de l'École polytechnique*. We can, therefore, sympathise with Gergonne's

7. Source from Lamy, Loïc "Le journal de l'École Polytechnique de 1795 à 1831, journal savant, journal institutionnel", *Sciences et techniques en perspective*, 32 (1995), 3-96.
8. From 1794, he was put in charge, alongside Monge, of the department of descriptive geometry at the École polytechnique, Hachette had Poisson, François Arago and Fresnel as students.
9 Dody, Brigitte, "La correspondance sur l'École polytechnique 1804-1816: un journal scientifique multidisciplinaire au service d'une école", *Sciences et techniques en perspective*, 28 (1994), 24-178.

assessment when he laments "The exact sciences, now developed so universally and so successfully, does not yet have one single periodical collection which is especially devoted to them", or when he adds:

> *As a result, the Journal de l'école Polytechnique, nor Mr Hachette's Correspondance can be considered as such: They are probably very valuable collections but, in addition to only appearing very irregularly, are devoted almost exclusively to the work of one institution* [10].



JOURNAL POLYTECHNIQUE,

OU

BULLETIN DU TRAVAIL

FAIT

A L'ÉCOLE CENTRALE DES TRAVAUX PUBLICS,

PUBLIÉ

PAR LE CONSEIL D'INSTRUCTION ET ADMINISTRATION DE CETTE ÉCOLE:

PREMIER CAHIER.

MOIS DE GERMINAL.

A PARIS,

DE L'IMPRIMERIE DE LA RÉPUBLIQUE.

AN III. 1794

Et se trouve chez les citoyens REGENT et BERNARD, libraires, quai des Augustins, n.° 37.

***Figure 3:* The front cover of the first issue of** **Journal de l'École polytechnique** *(it only took this name in the second volume), Germinal, year III (April 1795) (digitised by The National Library of France- BnF)*

Therefore, we can also understand the enthusiasm that the *Annales* aroused among mathematics teachers and their students, as well as students or former students of the École polytechnique. They also accounted for the vast majority of the authors of the contributions to the newspaper in its first years of publication. Teachers even encouraged their students to publish in the *Annales* de Gergonne. Their articles often bore the words "student of…" which was a way for them a way to make the quality and the level of their teaching known. The excellent standard of these articles and the novelty which they represented is therefore not surprising.

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10. Gergonne, quoted from a BibNum text. The document whose extracts we quote here was in fact the foreword of the first issue in which Gergonne and his collaborator Thomas Lavernède (Like Gergonne, he was also a Professor at Nîmes, he only participated in the writing of the *Annales* for two years) declared their intentions, the scope of the journal and the editorial line.

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Galois's article was in keeping with this "tradition" which had already been established for almost twenty years. The study of continued fractions was a subject which interested a large part of the mathematics community of that period. Galois quotes the work of Lagrange in the introduction to his article. The latter was the reference for all the works of algebra as well as analysis; the *Annales de Gergonne* was packed with articles (particularly on differential calculus) that used Lagrange's approaches and which constantly paid tribute to him - just as our young mathematician also does.

The reference made by Galois to "Lagrange's method" refers to the *Traité de la résolution numérique des équations de tous les degrés avec des notes sur plusieurs points de la théorie des équations algébriques*. In this treatise, the first edition of which was published in Year VI, Lagrange synthesised work and put it within the reach of every mathematician. He had begun this work almost thirty years before and started to distribute it in 1770 with his *Nouvelle méthode pour résoudre les équations littérales par le moyen des séries* parue dans les *Mémoires de l'Académie royale des sciences et belles lettres de Berlin* (T. XXIV, 1770).

The work of Lagrange and of Euler opened up many possibilities for development as did the *Annales de Gergonne*. Gergonne first published an update and new findings on the subject in 1819 in an article entitled: "Recherches sur les fractions continues[11]". Without going into detail about this article, it is interesting to note the observation made by the author in the introduction to his essay:

11. *Annales de mathématiques pures et appliquées*, IX (1818-1819), 261-270.

Malgré les travaux d'un grand nombre d'illustres géomètres, la théorie des fractions continues est loin encore d'être aussi avancée que son importance pourrait le faire désirer. Nous savons développer une fonction en fraction continue ; nous savons, dans quelques cas, revenir d'une fraction continue à la fonction génératrice ; nous savons aussi, dans quelques cas, reconnaître qu'une fraction continue est incommensurable ; mais personne encore n'a établi la limite précise qui sépare les fractions continues rationnelles de celles qui ne le sont pas. On ne saurait douter non plus que les fractions continues ne doivent affecter certaines formes particulières, suivant qu'elles sont racines d'équations de tel ou de tel autre degré, mais, passé le second degré, pour lequel nous savons que les racines se développent en fractions continues périodiques, nous ne connaissons plus les caractères qui distinguent les racines soumises à un pareil développement, ce qui serait pourtant d'autant plus important qu'à cette connaissance se rattacherait immédiatement la recherche des diviseurs commensurables de tous les degrés des équations numériques. Nous ne savons pas même former immédiatement la somme ou la différence de deux fractions continues, leur produit ou le quotient de leurs divisions ; et, à plus forte raison, ne savons-nous pas en assigner les puissances et les racines.

Despite the works of many renowned geometricians, the theory of continued fractions is still far from being as advanced as its importance warrants. We know how to convert a function into a continued fraction; we know, in some cases, to convert a representation of a continued fraction back into a generating function; we know also, in some cases to recognise that a continued fraction is incommensurable; but nobody has yet established the precise limit which separates rational continued fractions from those which are not. There can be no doubt either that continued fractions must not modify some particular forms depending on whether they are roots of equations of one degree or another, but beyond the second degree for which we know that the roots develop as periodic continued fractions, we do not know the characteristics that distinguish the roots subjected to such a development which would be even all the more important in view of the fact that this knowledge would be directly linked to the research of commensurable divisors of numerical equations of any degree. We do not even know how to immediately form the sum or the difference of two continued fractions, their product or the quotient of their divisions; and we do not even know how to find their powers and roots.

One of the most renowned geometricians who Gergonne talks about was Joseph-Louis Lagrange who undoubtedly deserves such recognition. This is why he often appeared in the contributions of the authors of the journal. Other articles on the topic followed, especially in Volume XIV (1823-1824)[12]. Gergonne refers to these articles in a footnote at the end of Galois's article. He also mentions an article published "in the present collection" but without quoting the author of it. It is probably in reference to an article written by himself: "Note sur un symptôme d'existence de racines imaginaires, dans les équations algébriques"[13]. In another footnote, he mentions a letter from the young Dupré,

12. We are quoting "Sur le développement en fractions continues des racines des équations numériques du second degree" [*Annales de Gergonne*, XIV (1823-1824), 324-333] whose anonymous author (the article is signed M***) is most likely Gergonne himself. Lagrange's method is mentioned in this article too. The article that follows is entitled "Sur le calcul des fractions continues périodiques" [*Annales de Gergonne*, XIV (1823-1824), 337-347] and is signed by the same M**.
13. *Annales de Gergonne*, XIX (1828-1829), 124-126.

a "distinguished student of the École normale and of collège Royal de Louis-Le-Grand". One year ahead of Galois in this school (then a "preparatory class"), Dupré is another example of those students who were encouraged by their teachers to publish in the *Annales*. The year before, he wrote an article which Gergonne also refers to and from which he practically borrows the title: "Note sur un symptôme d'existence de racines imaginaires, dans une équation de degré"[14]. We now have a better idea of the mathematical and editorial context in which the young Galois is driven to publish his article on continued fractions[15].

GALOIS'S PROOF OF A THEOREM ON PERIODICAL CONTINUED FRACTIONS

It is worth reminding ourselves about continued fractions in order to get a better understanding of the gist of his article. Let's take the equation: $x^2 + x - 1 = 0$. It is the same as $x(x + 1) = 1$. Hence $x = \frac{1}{1+x}$

In the right-hand member, if we replace x by $\frac{1}{1+x}$, we get: $x = \cfrac{1}{1+\cfrac{1}{1+x}}$

There is nothing to stop us from starting again and again. So x is written as a fraction which does not stop – we call this a *continued* fraction:

$$x = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}}}$$

In his *Introduction à l'Analyse infinitesimale,* translated into French in 1796, Euler describes this concept[16] :

> *I call a continued fraction a fraction whose denominator is the sum of an integer and a fraction, whose denominator is the sum of an integer and a fraction formed in the same way as the previous ones. This type of process can proceed indefinitely or it can stop.*

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14. *Annales de Gergonne*, XVIII (1827-1828), 68-71]. Gergonne writes: "by M. A. Dupré, student of the preparatory school of Collège de Louis-le-Grand"

15. The following year, Galois published another article (he had begun the preparatory class of Lycée Louis le Grand, that is, the École normale) entitled : "Notes sur quelques points d'analyse" [*Annales de mathématiques pures et appliquées*, XXI (1830-1831), 182-184]. This article provides an extended proof of the existence of the derivative function (still Lagrange), and an interesting result on the radius of curvature of curves in space. It was studied by Massimo Galuzzi: "Galois' Note on the Approximative Solution of Numerical Equations (1830)", *Archives for History of Exact Sciences*, 56 (1) (2001), 29-37.

16. *Introduction à l'analyse infinitésimale*, "traduite du latin en français, avec des Notes & Éclaircissements", by J.B. Labey, first volume published by Barrois, 1796 (page 277).

The previous fraction only consists of blocks of 1. More generally speaking, we say that a continued fraction is periodic if blocks reappear. Here they are blocks of one. If they are blocks of four terms, the fraction is of the form:

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{a+\ldots}}}}}}}}$$

In the interests of brevity, we will write down such a fraction [a,b,c,d]. This will save us from writing the expression of a general periodic fraction: [a_0, a_1, a_2, . . ., a_n].

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The young Galois proves the following result:

If one of the roots of an equation of any degree is an immediately periodic fraction, this equation will necessarily have another root which is also periodic which is obtained by dividing the negative unity by this same periodic continued fraction, written in the reverse order.

To prove his result, he restricts himself to a period of four terms because, as he explains,

The uniform progression of the calculus proves that it would be the same if we allowed a greater number.

In other words, he proves that if [a,b,c,d] is the root of an equation, it will necessarily have −1/[d,c,b,a] as another root. The demonstration is simple although somewhat tedious to write. To make the writing process easier, we have done it with two terms using Galois's approach but he is happy to do it for four terms.

Galois's proof on the theorem on continued fractions, but with two terms

Galois's result means that, if $x = a+\cfrac{1}{b+\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{a+\dots}}}}$ is the solution to an equation, then $y = -\cfrac{1}{b+\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{a+\dots}}}}$ is also the solution to the equation.

Because, if $x = a+\cfrac{1}{b+\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{a+\dots}}}}$; this means that $x = a+\cfrac{1}{b+\cfrac{1}{x}}$.

So:

$$x-a = \cfrac{1}{b+\cfrac{1}{x}} \Leftrightarrow (x-a)\left(b+\frac{1}{x}\right) = 1 \Leftrightarrow bx^2 - abx - a = 0$$

By doing the same with the second fraction:

$y = -\cfrac{1}{b+\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{a+\dots}}}}$ **Erreur ! Signet non défini.** $b+\frac{1}{a-y} = -\frac{1}{y}$

Erreur ! Signet non défini. $by^2 - aby - a = 0$

We arrive back at the same equation. So, the two (continued) fractions are solutions to the same equation.

AN EXAMPLE OF AN EQUATION PROCESSED BY GALOIS

Galois devotes the end of his article to a numerical application consisting of the study of the equation: $3x^2 - 16x + 18 = 0$ *[equation. (1)]*. He remarks that one of the roots is between 3 and 4 (which is deducted immediately since when x=4 the number is positive and when x = 3 it is negative).

He applies Euler's method which is based on the integer part of one of the roots. This is similar to the expression of a rational number in the form of a continued fraction (see box).

Continued fractions to express rational numbers[17]

Any rational number can be written as x = a + 1/y, *a* being an integer and *y*>1; we then repeat the process which corresponds to the division by the Euclidean algorithm. So, if we simplify x = 314159/100000. We will write successively:

$$100000 = 7 \times 14159 + 887$$
$$14159 = 15 \times 887 + 854$$
$$887 = 1 \times 854 + 33$$
$$854 = 25 \times 33 + 29$$
$$33 = 1 \times 29 + 4$$
$$29 = 7 \times 4 + 1$$

Euclidean's algorithm ends and the same thing happens for any rational number. So:

$$x = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{25 + \cfrac{1}{1 + \cfrac{1}{7 + \cfrac{1}{1}}}}}}} = [3,7,15,1,25,1,7,1]$$

Therefore, Galois tries to locate the positive root more accurately by writing down[18] x = 3 + 1/y. He then gets a transformed equation: $3y^2 - 2y - 3 = 0$ *[equation. (2)]*. This is exactly the type of equations written as $ax^2 - bx - a = 0$ that Galois studied in the previous pages and for which he proved had immediately periodic roots; one necessarily greater than 1 and the other contained[19] between -1 and 0. What we mean by *immediately* periodic is a continued fraction whose period starts with the first term.

Galois then processes the equation $3y^2 - 2y - 3 = 0$. He writes down, still using the same method as the integer part of the root (we have seen that the positive root of equation *(2)* is greater than 1 but less than 2): y = 1 + 1/z. Replacing in equation *(2)* in *y*, he gets $2z^2 - 4z - 3 = 0$. The positive root of this equation being greater than 2 but smaller than 3, so he writes down z = 2 + 1/t and gets the transformed equation $3t^2 - 4t - 2 = 0$. The positive root of this

17. This box is extracted from the Bibnum text by Alain Juhel who comments on a text by Lambert (1761), and who uses continued fractions extensively to prove the irrationality of π.
18. Here, there is a typo in the text: *x = 3x + 1/y*. Liouville fixed this typo in its republication in 1846.

equation is greater than 1 but smaller than 2, so he writes down t = 1 + 1/u and gets the transformed equation $3u^2 - 2u - 3 = 0$. The identity of the equation in *u* and of equation *(2)* in *y* proves that the continued fraction forming *y* is immediately periodic and applies if we go back to *y* as:

$$y = 1 + \frac{1}{z} = 1 + \cfrac{1}{2 + \cfrac{1}{t}} = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{u}}} = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{y}}}.$$

By substituting *y* on the bottom right-hand side with the whole expression that makes up the last right-hand member, we get the immediately periodic continued fraction:

$$y = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{y}}}}}} = [1,2,1,...]$$

Galois was therefore able to use this example to show the important result that he had discussed pg.299:

> *Therefore, when we process a numerical equation using the Langrangian method, we can be sure that there will be no periodic roots as long as we do not find a transformed equation having at least one positive root greater than the unity and another contained between 0 and -1; and if, indeed, the positive root that we are looking for must be periodic, it will be, at most, at this transformed equation that the periods will begin.*

So, equation *(2)* in *y* is an example of an equation from which the period of the continued fraction begins. This is not the case with equation *(1)* in *x*: it is necessary to wait for the first transformed equation in *y*, equation *(2)*, for the period to begin. Equation *(2)* shows another property that Galois had discussed:

> *Any second degree equation of the form $ax^2 - bx - a = 0$ will have both immediately periodic and symmetric roots.*

The solution of equation *(2)* is immediately periodic, but as it is of the form [1,2,1,…], it is also symmetric. When we reverse the period, we find [1,2,1,…]. In accordance with the theorem proven by Galois at the beginning of the article, the roots of equation *(2)* are then [1,2,1,…] and -1/[1,2,1,…]. In this particular

19. We can easily check this position of the roots by taking the values for -1 (positive), for 0 (negative), for 1 (negative) and for the infinity (positive). Remember that in the equations set out by Galois, a and b are positive integers.

case of symmetry of the continued fraction, the equation has two roots which are *A* and -*1/A*[20].

Galois finally gives the form of the solutions of the initial equation *(1)*. Unlike that of *(2)*[21], they do not form immediately periodic continued fractions:

les deux valeurs de x seront donc

$$x = 3 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\ldots}}}}}}, \qquad x = 3 - \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\ldots}}}}}}.$$

Again, Galois or perhaps even the publisher is mistaken. The left value of *x* is correct but the right value is not. The author got mixed up and took x = 3 + y instead of x = 3 + 1/y. However, it must be said that with two values of *y,* one of which is equal to the opposite of the reverse of the other, this type of mistake can easily happen. Equation *(1)* in *x* consists of two roots; one contained between 3 and 4 (value at the left in the above figure) and the other contained between 1 and 2 (value at the right-hand side). Mistakenly, this latter value is contained between 2 and 3. Even though he rectifies others, this error also got past Liouville in his 1846 edition. The correct right value of *x* is[22] :

$$x = 3 + \frac{1}{y_n} = (3 - y_p) = 2 - \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\ldots}}}}$$

However, we should give Galois the benefit of the doubt since the end of his article develops as though he had written the correct value of *x* above. In fact, he suggests another way of writing this value by using the following identity:

$$p - \frac{1}{q} = (p-1) + \cfrac{1}{1+\cfrac{1}{q-1}}$$

20. We can use the method of discriminants to check that the equation $3y^2 - 2y - 3 = 0$ has two roots, $1/3(1+\sqrt{10})$ and $1/3(1 - \sqrt{10})$, one of which is the opposite of the reverse of the other.
21. Note again, an error in the original article at the top of page 301; "the positive value of y" begins with a negative sign. Also, the succession of numbers does not start correctly.
22. Here we called y_p the positive root of equation (2) in *y*, and y_n its negative root, knowing that $y_n \times y_p = -1$.

By applying this identity to the value of *x* above, with *p* = 2, we get the value given at the very end of Galois's article; a value which is correctly contained between 1 and 2):

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \ldots}}}}}}}}}$$

One may wonder why he does this at the end. Perhaps he considered it more elegant to represent continued fractions using only positive signs.

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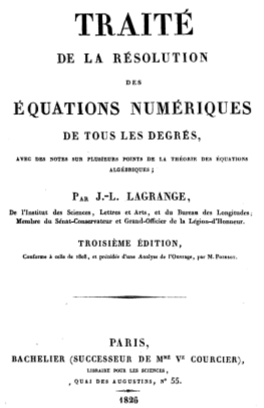
In any case, it is clear that in order to situate Galois's first article better still, it would be worth closely studying the collection made up of all the articles relating to continued fractions published in the *Annales de Gergonne* or in other texts. Without having an exhaustive study at our disposal, all of these works seems to be full of commentaries on the work of Lagrange on this subject. Lagrange's *Traité de la résolution des équations numériques de tous les degrés* had just been republished in a new edition in 1826 by Bachelier. It is probably this edition that Galois consulted. The works of Lagrange were distributed in hundreds of copies and were texts of reference of which Galois and many others had been readers, and to a certain extent, successors.



TRAITÉ
DE LA RÉSOLUTION
DES
ÉQUATIONS NUMÉRIQUES
DE TOUS LES DEGRÉS,
AVEC DES NOTES SUR PLUSIEURS POINTS DE LA THÉORIE DES ÉQUATIONS ALGÉBRIQUES;
PAR J.-L. LAGRANGE,
De l'Institut des Sciences, Lettres et Arts, et du Bureau des Longitudes; Membre du Sénat-Conservateur et Grand-Officier de la Légion-d'Honneur.
TROISIÈME ÉDITION,
Conforme à celle de 1808, et précédée d'une Analyse de l'Ouvrage, par M. Poinsot.

PARIS,
BACHELIER (SUCCESSEUR DE Mme Ve COURCIER),
LIBRAIRE POUR LES SCIENCES,
QUAI DES AUGUSTINS, n° 55.
1826

***Figure 4 :* Traité de la résolution des équations numériques de tous les degrés by J.L. Lagrange.** *The third edition (1826) of the* Traité de la résolution numérique *is*

true to the 1808 edition which is made up of Lagrange's memoirs published in Recueil des mémoires de l'Académie de Berlin *(1767 & 1768) added to which were various notes.*

A MATERIAL READING OF GALOIS'S TEXT

Galois's text is also worth a material reading. Let's draw a comparison between Galois's two texts, published firstly in the *Annales de Gergonne* in 1828-1829 and then in the *Journal de Liouville* in 1846[23].

Both texts were printed by the same publisher fifteen years apart and the same words and the same formulas were used. However, the latter edition does not use the same typographical object and this is what makes the articles so different from each other.

A comparison of Galois's two "identical" texts published in 1828-1829 and then in 1846

Here are two examples of different printings in the original article of 1828 and in the republished version of 1846 (*Journal de mathématiques pures et appliquées*, I, 11 (1846), pg.385-392).

bre. Soit une des racines d'une équation de degré quelconque exprimée comme il suit:

$$x=a+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\ldots\ldots;$$

des racines d'une équation de degré quelconque exprimée comme il suit :

$$x=a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\ldots}}}}}}};$$

23. For the circumstances of the publication of *Œuvres de Galois* in the *Journal de Liouville* we refer to the work of Caroline Ehrhardt: "La naissance posthume d'Évariste Galois", *Revue de synthèse*, 131, 6th series, 4 (2010), pg. 543-568 ; also *Évariste Galois. La fabrication d'une icône mathématique*, Les Éditions de l'EHESS, 2011, pg. 185-193.

[Above, BibNum text, pg. 295] [Below, 1846 text, pg.386]
In addition to the written form of the continued fraction, we can also notice the position of the semi colon in both cases.

proposée ne pourra être de la forme $x=p+\frac{1}{A}$, car alors , en vertu de notre théorème, la seconde devrait être $x=a+\cfrac{1}{-\cfrac{1}{B}}=a-B$;

car alors, en vertu de notre théorème, la seconde devrait être

$$x=a+\cfrac{1}{-\cfrac{1}{B}}=a-B;$$

[Above, BibNum text, pg. 298] [Below, 1846 text, pg.389]
The "on-line" equation (x=a+…), which is difficult to understand in the 1829 (even though it is not a continued fraction), becomes an "off-line" equation in 1846.

The manner in which continued fractions were represented – composed of a complex entanglement of fractions – progressed considerably between 1829 and 1846. By 1846, typographers could manage many fractions written with different sized symbols while still making them look aesthetical and harmonious. In 1846, the mathematical press entered another phase of its development. It became professional in the sense that, henceforth, typographers were trained to represent mathematical symbols in a clear manner. These included the representation of fractions, of exponents, of suffixes, of charts, of summation signs and so on. This is also what made it possible to confirm the identical nature of Galois's two texts on this theorem of continued fractions.

It is with this strategy of comparison of the identical texts that we have been able to compare characters that are difficult to decipher such as the integral sign with different limits and sizes depending on the form of the integrand, the summation sign or the different geometric expressions. By comparing both examples, we believe to have shown that Bachelier - the publisher of the first French mathematical journals - was able to take a considerable lead over its main competitors by making the clear representation of mathematics one of its main focuses for innovation[24].

■

(December 2011)
(V2 with appendix January 2012)
(translated in English by Lauren Gemmell, published September 2013)

24. "Vendre et éditer des mathématiques avec la maison Bachelier (1812-1864)", *Revue d'histoire des mathématiques* [submitted article].

Appendix
Some variations on continued fractions

In any second degree equation, $Tx^2 + Ux + V = 0$, we know that the product of the roots is V/T[25]. In the particular equation mentioned by Galois, $ax^2 - bx - a = 0$, the product of the roots is -1. Therefore they are reversed and opposed, A and -1/A which Galois demonstrates using continued fractions.

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Let us now consider an even more special case than that one: the golden ratio equation $x^2 - x - 1 = 0$. We have:

$$x = 1 + \frac{1}{x} = 1 + \frac{1}{1 + \frac{1}{x}} = \ldots = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ldots}}}} = [1,1,1,\ldots]$$

It is important to note that the series of equal signs before the ellipsis is valid for the two roots of the equation (positive and negative, the reverse and opposite of the other). When we write the final equal sign, we place emphasis on the *positive root* because [1,1,1,…] is a positive number. There is an implied passage to the limit between what an equation is (to the left of the ellipsis) and what becomes a number (to the right of the ellipsis), which is represented by a continued fraction. It is in this sense that the processing of infinite continued fractions requires some precaution.

In order to find the expression in continued fraction of the negative root, we do it differently. We write[26] :

$$x = -\frac{1}{1 - x} = -\frac{1}{1 + \frac{1}{1 - x}} = \ldots = -\frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ldots}}}} = -[0,1,1,1,\ldots]$$

This result is easier to find using Galois's theorem (reversed and opposite roots): A (positive root) = [1,1,1,…], so B (negative root) is as such[27]:

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25. To prove this, we develop $T(x-x_1)(x-x_2)$, where x_1 and x_2 are the roots. These are the classic connection between the roots and the coefficients of the equation which Galois focused on in his theory of equations.
26. The 0 at the beginning of the square brackets is important as it reminds us that there is no term before the fraction (unlike the positive root).

$$B = -\frac{1}{A} = -\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}}} = -[0,1,1,1\dots]$$

@@@@@@@

Let's do an exercise which is a bit different now by focusing on C = [0,1,1,1,….] = – B, a number contained between 0 and 1. This number has the distinctive feature of forming 1 when added to its square: $C^2 + C = B^2 - B = 1$ (because B is the solution to the equation $x^2 - x - 1 = 0$). We know the continued fraction representation of C but what is that of C^2?

Correspondence with real numbers - Continued Fraction

For those who struggle with the representation of a number as a continued fraction, we will give the clear correspondence with *real* numbers. By doing so we can return to *reality*. However is this representation not similar to another?

A = (1+√5)/2 = 1,618… [golden ratio]
B = (1- √5)/2 = - 0,618…
C = (√5 - 1)/2 = 0,618…
C^2 = (3 - √5)/2 = 0,381…

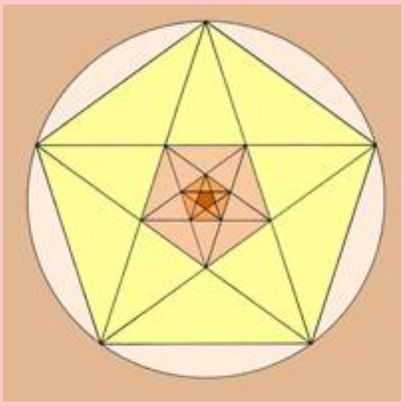

***Figure 5*: Convex pentagons and pentagrams within each other.** *The diagonal of the convex pentagon (this is also the side of the pentagram) to its side is equal to the golden ratio.*

Let's look for two ways of representing C^2. In the first case, let's suppose that we do not know the development of C as a continued fraction. So let's look for an equation of which C^2 is the solution. C is the solution of the equation C^2 +

27. We put the expression of A as a continued fraction in blue for more clarity.

C – 1 = 0. Let's group together odd powers on one side and even powers on the other and then let's square:

$(1 - C^2) = C \qquad (1 - C^2)^2 = C^2 \qquad C^4 - 3C^2 + 1 = 0$

So C^2 is the solution to the equation $x^2 - 3x + 1 = 0$; one of the solutions of this equation is contained between 2 and 3. We will then apply Galois's method (see above) by writing down $x = 2 + 1/y$. The transformed equation becomes $y^2 - y - 1 = 0$ and enters into the configuration of $ax^2 - bx - a = 0$ equations and has [1,1,1,....] and – [0,1, 1,...] for roots (it turns out that it is the equation of the golden ratio: see above). Now C^2 is a number contained between 0 and 1 (since $C + C^2 = 1$), so we will choose the negative value of *y* to add to 2:

$$C^2 = 2 + \frac{1}{y} = 2 + \frac{1}{-\left(\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}\right)} = 2 - \left(1 + \cfrac{1}{1+\cfrac{1}{1+\ldots}}\right) = 1 - \cfrac{1}{1+\cfrac{1}{1+\ldots}}$$

By using Galois's formula, $p - \frac{1}{q} = (p-1) + \cfrac{1}{1+\cfrac{1}{q-1}}$ (here $p = 1$), we write (q is in blue:

$$C^2 = 1 - \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}} = 0 + \cfrac{1}{1+\cfrac{1}{\left(1+\cfrac{1}{1+\cfrac{1}{1+\ldots}} - 1\right)}} = \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\ldots}}} = [0,2,1,1,1\ldots]$$

This development as a continued fraction can be checked by a second method - this time by using the development of C:

$$C^2 = 1 - C = 1 - \cfrac{1}{1+\cfrac{1}{1+\ldots}}$$

This brings us back to the previous calculation and therefore to the same result. Or, even simpler still, without using Galois's formula (necessary to reduce the negative signs):

$$C^2 = \left(\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}}\right)^2 = \frac{1}{(1+C)^2} = \frac{1}{(C^2+C)+(C+1)} = \frac{1}{2+C} = \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}}$$

@@@@@@@

We then arrive at the following rather curious identities knowing that C + C² = 1 :

$$\left(\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}}\right)^2 = \left(\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}}\right)$$

$$\left(\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}}\right) + \left(\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}}\right)^2 = 1$$

$$\left(\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}}\right) + \left(\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}}\right) = 1$$

■