Leibniz’s *De Progressione Dyadica* manuscript

by Yves Serra, engineer

Bibliographies on Leibniz all refer to two publications that are important to the history of calculation and calculating machines. One of these, published in 1710 in the *Miscellanea Berolinensis*, presents the arithmetic machine[1] that Leibniz designed between 1673 and 1709. The other, published in 1703, is a paper for the Paris Academy of Sciences on binary calculation (“An Explanation of Binary Arithmetic Using Only the Characters 0 & 1, with Remarks about its Utility and the Meaning it Gives to the Ancient Chinese Figures of Fohy[2]”).

By comparing these two publications, one would hope to find a text, which, by summarising binary calculation and mechanisation, stands as a precursor to modern-day computing. But nothing of the sort can be found in the official bibliographies nor in other publications, and the question draws a blank among Leibniz specialists, who are often more interested in the philosophical dimension of the author’s work than in his technical inventions. Might Leibniz have trodden the two paths without ever seeing them meet? This analysis resumes the investigation and leads us to the satisfying conclusion that, no, Leibniz did not overlook this possible rapprochement and that there does indeed exist a manuscript, dating from 1679, which, at present, appears to be the earliest known description of a binary calculator.

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The inquiry begins with various Internet searches on French, German, Latin and English search engines. After a great deal of trial and error, this produces references to a text, still in manuscript form, which was never published by Leibniz or included in his works. It therefore appears to have gone largely unnoticed by the commentators. The manuscript is dated 15 March 1679. It is entitled *De Progressione Dyadica* and is held at Hanover Library (Niedersächsische Landesbibliothek Hannover). The manuscript is so little known

1. See the following analysis on *BibNum* (Yves Serra, August 2009).
2. In French, *Explication de l’arithmétique binaire qui se sert des seuls caractères 0 & 1 avec des remarques sur son utilité & sur ce qu’elle donne le sens des anciennes figures chinoises de Fohy*. Text online here.

that an authoritative work on Leibniz and binary notation simply does not mention it.[3] A let-down.

With only the title to go on (*De Progressione Dyadica*), one naturally assumes that the manuscript must foreshadow the paper of 1703. What remains to be seen is how it presents mechanised calculation. Evidence is required.

Unfortunately, the original three-page Latin manuscript is not available on the Internet. Nor is any translation. So one turns to less electronic sources and, eventually, the research converges on a work in German published in 1966, which is cited by several researchers:

> Leibniz, De Progressione Dyadica, Pars I, (MS, 15 March 1679), published in facsimile (with German translation) in Erich Hochstetter and Hermann-Josef Greve, eds., *Herrn von Leibniz' Rechnung mit Null und Einz* (Berlin: Siemens Aktiengesellschaft, 1966).[4]

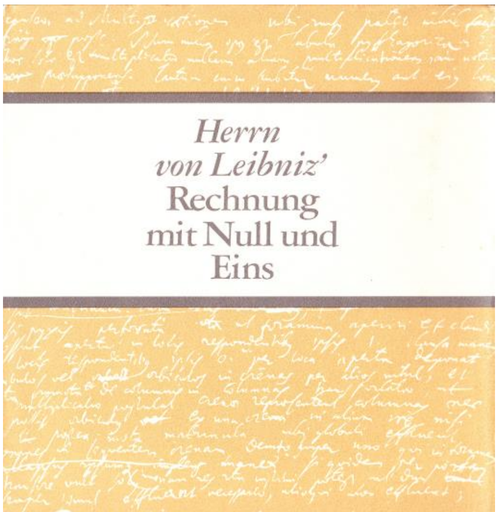

Figure 1**: The book (published in 1966) in which Leibniz's 1679 manuscript,* **De progressione dyadica, was published for the first time.* *The manuscript is kept at the Niedersächsischen Landesbibliothek Hannover.*

Now one needs to get hold of the book. It transpires that it can be obtained over the Internet from an antiques seller in Munich, who has a copy. The book does indeed contain a three-page facsimile of Leibniz's manuscript, a German translation by the clergyman Pater Franz X. Wernz S. J. (Munich), and an analysis, in German, written by Hermann-Josef Greve.[5] After a few months of investigations, work can finally begin! And, on its third page, the manuscript

3. H. J. Zacher, *Die Hauptschriften zur Dyadik von G. W. Leibniz*, Klosterman, Frankfurt, 1973.
4. The work was published by Siemens in 1966 to mark the 250th anniversary of Leibniz's death (14 November 1716). The English translation of the title is: *Calculation using 0 and 1, according to Herr von Leibniz*.
5. Hermann-Josef Greve, born in 1903, a graduate of the University Cologne, director of Saarbergwerke AG, Saarbrücken.

does indeed contain what is doubtless the first description of a binary calculator. In his analysis, Hermann-Josef Greve pays tribute to Eric Hochstetter[6] for having discovered this description in this manuscript by Leibniz housed in Hanover Library – and *he* didn't have the Internet to help him!

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As the reader will have grasped, the aim of this lengthy introduction is to show how difficult it is, even today, to trace original texts, even if they are essential and even if they are written by authors as renowned as Leibniz. At the same time, it also underlines the usefulness of initiatives like *BibNum*, which are not content to publish easily accessible content for the nth time.

Leibniz's manuscript is reproduced here alongside a French translation produced by the author of this article. The translation draws on both the Latin text and the version in German, but above all on the binary content in Leibniz's examples!

LEIBNIZ'S DESCRIPTION OF BINARY NOTATION

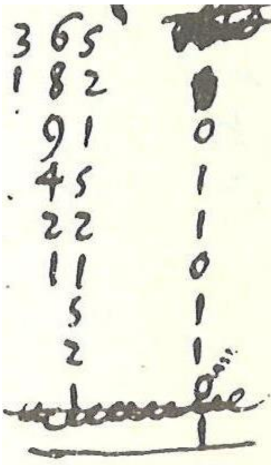
The first page describes binary notation. Referring to a list of the first 100 numbers written in base 10 and base 2, Leibniz first explains how to move from one binary number to the next. He explains the calculation using successive powers in the base, using as an example the transition from 1011000 to 88.

For the inverse conversion from the decimal to the binary scale, Leibniz suggests dividing the given number successively by 2 and noting down the remainder, i.e. either 0 or 1 at each stage. Thus 365[7] becomes 101101101 (fig. 2).

He generalises this method by showing that by dividing this same binary number 101101101 by 1010, i.e. by 10 in the decimal scale, and by retaining the remainders at each division, this takes us back to 365. The division is performed here using the method *à la française*, which is discussed below alongside Leibniz's explanations of binary division.

6. Erich Hoschstetter (1888–1968), professor of philosophy at the University of Münster, director of the Münster Leibniz Research Centre and a corresponding member of the Göttingen Academy of Science.
7. Leibniz often uses the number 365 (days of the year) as an example in his articles. See the *BibNum* text on Leibniz's machine, in which Leibniz uses as an example the multiplication 365 x 1709 (the year he wrote his article).

The operation consists in two divisions: the first is 101101101 (365) divided by 1010 (10), which gives a quotient of 100100 (36) and a remainder of 101 (5); the second is 100100 (36) divided by 1010 (10), which gives a quotient of 11 (3) and a remainder of 110 (6).[8]



365	
182	1
91	0
45	1
22	1
11	0
5	1
2	1
1	0
	1

***Figure 2*: Leibniz's text (left) and its transcription (right).** *Dividing 365 by 2 gives 182 and this leaves 1, which is noted in the right-hand column; dividing 182 by 2 gives 91 and this leaves 0, which is noted in the right-hand column, etc. The final 1 (bottom left) is systematically transferred to the bottom right. Note: The binary number is read from the bottom to the top, so 365 is written 101101101 in base 2.*

The first page also contains two annotations that are separate from the main text. The first is a comment on the patterns in the string of 0s and 1s in the succession of natural numbers expressed in the binary scale. In this annotation, Leibniz sets out a plan to find other patterns in the progressions of figures in binary notation.

The second annotation stresses how easy binary multiplication is to use and how it does away with the need to learn the times table. This point deserves to be emphasised, given what follows.

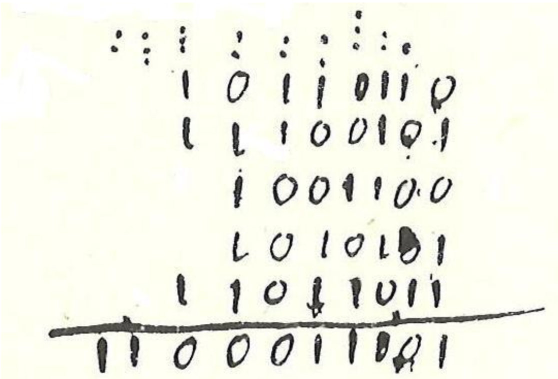
8. The classic method of binary division (and not the French style of division that Leibniz uses) needs to be adapted here. The first division, 101101101 divided by 1010, does not pose any problems, because it works in the same way as in the decimal system: we can easily calculate the quotient (100100) and the remainder (101). The division of this quotient, 100100, by the divisor, 1010, is a little trickier (reminder: we are using only 0s and 1s): in 10010 there is one 1010 and a remainder of 1000. We put down one 0 (the last number in the dividend). In 10000 there is one 1010 and a remainder of 110. This second division therefore gives 11 as the quotient and 110 as the remainder.

Binary addition and subtraction

On the second page of the manuscript, Leibniz begins by describing binary addition. He confirms that this type of addition is very easy to use, observing that after noting down 10110 (and then 11011 underneath), he can directly note down the answer.

Unfortunately, he writes 1000001 instead of 110001.[9] The other more complex – and correct – additions that Leibniz discusses later on indicate that this is simply a mistake and, above all, that this is an unread manuscript which Leibniz would never have published in its current state.

After this first simple addition, Leibniz moves on to an addition of five numbers, without any mistakes this time. He explains that binary addition simply entails adding up the 1s in each column. There is no need to know the addition table, unlike in calculations using decimal notation. Continuing his demonstration, Leibniz suggests combining additions and subtractions and working out the answer by simply offsetting the +1s and -1s in each column.



```
  3 2 2 2 3 2 1
    1 0 1 1 0 1 1 0
    1 1 1 0 0 1 0 1
      1 0 0 1 1 0 0
      1 0 1 0 1 1 1
    1 1 0 1 1 0 1 1
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1 1 0 0 0 1 1 0 0 1
```

Figure 3: Leibniz's text (left) and its transcription (right).
Starting in the right-hand column, we count three 1s, write 1 at the bottom and carry over 1; with the remainder, the second column gives four; we set down 0 and carry over 2; with the remainder, the third column gives six; we set down 0 and carry over 3 (as Leibniz writes, if the number of 1 is even, we "transfer half the number of units to the following column"; etc. The top of our table shows the successive remainders, which Leibniz marks as dots.

Finally, Leibniz presents the technique of subtraction by adding the tens' complement, a technique he had already presented in the decimal scale and which was also used by Pascal to carry out subtractions with his "Pascaline".[10]

9. In decimal notation, this is 22 + 27 = 49. In binary notation, Leibniz makes his mistake in the last column on the left: the correct approach is to add 1 and 1, and there is already a remainder of 1 from the other columns. We therefore need to write 11 at the bottom (whereas Leibniz, in his enthusiasm, inadvertently writes 100).
10. See this *BibNum* analysis of Blaise Pascal's text on his machine (1645), by Daniel Temam (March 2009).

BINARY DIVISION AND MULTIPLICATION

The top of the third page of the manuscript discusses multiplication and the binary calculator (which we will turn to at the end of this article), while the bottom part discusses division. Leibniz returns to the division he has already considered on the first page and transforms 365 from binary notation (101101101) to decimal notation using successive divisions by 10 (or 1010 in the binary scale). To do this, Leibniz uses the "French" method of division. This consists in setting down the dividend and, below that, the divisor, as many times as there are powers in the base. To the right of the dividend, we construct the quotient by gradually setting down the successive remainders above the dividend, from left to right, and striking out the numbers that have already been used. The examples below, which are expressed in the decimal scale and become progressively become difficult, are taken from the book *L'arithmétique en sa perfection* by F. Le Gendre (Paris, 1684):

36 / 4 = 9 becomes

~~36~~ ∫ 9
~~4~~

8785 / 5 = 1757 becomes

~~323~~
~~8785~~ ∫ 1757
~~5555~~

6754 / 357 = 18 with a remainder of 328 becomes

3
~~7~~
~~1~~2
~~328~~8
~~6754~~ ∫ 18
~~3577~~
~~35~~

The video in the link[11] shows how this last division is carried out. Using the same notation, the division of 365 by 10 becomes the following in binary notation:

11. The French method of division: http://yves.serra.pagesperso-orange.fr/division.htm (teaching video).

~~101101~~101 ∫ 100100

~~1010010~~

~~1~~

The quotient is therefore 100100, i.e. 36, and the remainder is 101, i.e. 5. This division is presented very simply, and Leibniz remarks that it can even be performed without striking out the numbers and noting down the intermediary stages. This simplicity is not in fact due to the binary numbering but to the absence of remainders in this example. Leibniz notices this no later than the following line, where, in order to once again divide 36 by 10, i.e. 100100 by 1010, he starts over three times, before crossing out his work and finally returning to the "French" method:

1

~~110~~1

~~10010~~0 ∫ 11

~~1010~~0

~~101~~

In other words, a quotient of 11, or 3 in decimal notation, and a remainder of 110, or 6 in decimal notation.

Following this example, Leibniz explains how to obtain a continuous remainder when subtracting. He presents this as a practical algorithm, which today we would write as

```
subtract 1
while above there is zero,
                write 1
                move one column to the left,
end while
above there is 1,
write zero
end
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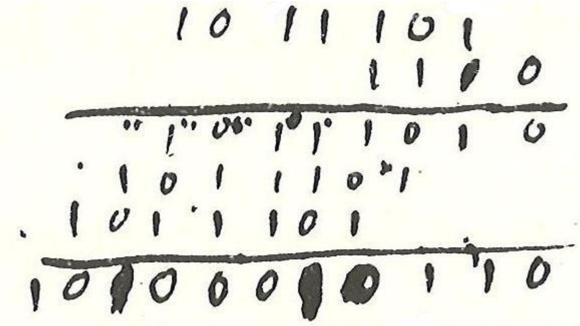
Let's now turn to the section of the manuscript which covers multiplication. Leibniz first emphasises the major contribution binary notation has made. No more times tables to learn! No more Pythagorean tables. To multiply one simply has to write the number itself or to shift it sideways one zero.

This principle is illustrated with an example, which we reproduce in full below because it enables us to visualise the idea Leibniz that goes on to develop:



			1	0	1	1	1	0	1	
							1	1	1	0
			1	0	1	1	1	0	1	0
		1	0	1	1	1	0	1		
	1	0	1	1	1	0	1			
1	**0**	**1**	**0**	**0**	**0**	**1**	**0**	**1**	**1**	**0**

***Figure 4*: Leibniz's text (left) and its transcription (right).** *The multiplication shown is 93 = 14 = 1302 (see explanations below).*

The following paragraphs contain much of what is original about this manuscript, i.e. mechanised calculation.

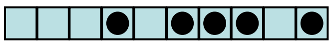
MECHANISING CALCULATION

In the very first line, Leibniz announces the possibility of a mechanical machine being able to effortlessly perform this multiplication in binary notation.

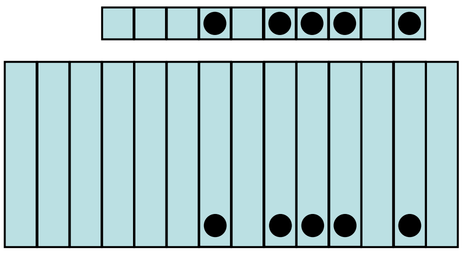
This type of calculation could also be performed with a machine (without wheels) … Using a box with holes that can be opened and closed.

Let's imagine, then, a box containing a series of holes that release balls in accordance with each line of the above multiplication:

1 0 1 1 1 0 1

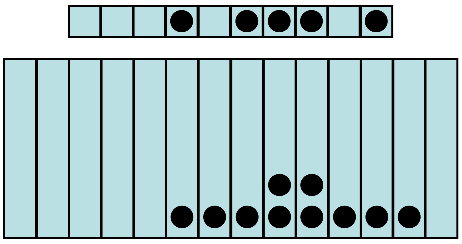


The balls then fall into grooves, which represent the columns in the multiplication:

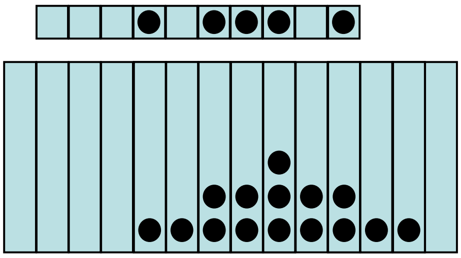


***Stage 1*:** *The multiplier 1110 contains a 0 to the right, so we leave a column to the right (just as when multiplying by hand). The second number from the right is a 1, so we stop there and release the balls.*

The box will be moved to the left as many times as is required by the multiplier, i.e. 3 times in this example of multiplication by 1110 (i.e. 14).



Stage 2: *Second shift, corresponding to the next number 1. We move another column to the left and release the balls, which pile on top of those already there.*



Stage 3: *Third shift, corresponding to the next and last number 1.*

Leibniz had also used this design (with a fixed and moving part) for his decimal calculating machine.[12] The first sketches of this machine date from 1673 – that is to say, prior to this manuscript – and it is likely that Leibniz reused the same idea here.

Leibniz then naturally point outs that the balls must not be able to move from one groove to another.

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We now need to consider how the balls in a same groove are used to calculate the total and remainders. Leibniz now formulates two key ideas. One: that the balls must "fall down" if there are two of them in a groove, whereas if there is only one ball in the groove, it must stay where it is:

> *For it can be organised so that two balls are necessarily released together, and otherwise are not released.*

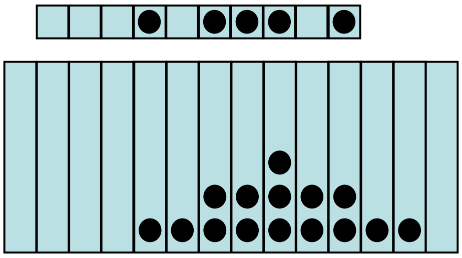
And two: that of the two balls which fall down, one will go into the next groove along and the other one will fall into a repository under the machine. The first ball represents the remainder.

It is clear that these two ideas are both necessary and sufficient to obtain the desired calculation, i.e. the total and remainder. Let's first take a look at

12. See this previously cited *BibNum* text.

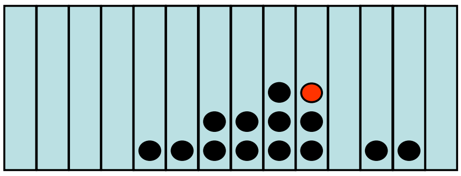
what happens visually before returning to the mechanical aspects Leibniz sketches out.

The moving box (shown above) has released balls three times. This corresponds to the multiplication of 1011101 by 1110, i.e. the result shown in stage 3 above:

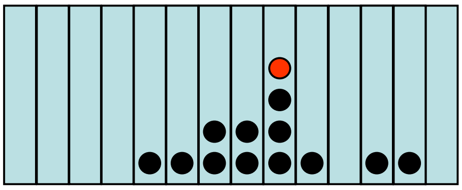


The fourth stage consists in calculating the result from right to left. In the unit column there are zero balls, so the result is 0. In the next two columns along, corresponding to 10 and 100, i.e. 2 and 4, there is only one ball, which, following the first rule, stays in its groove.

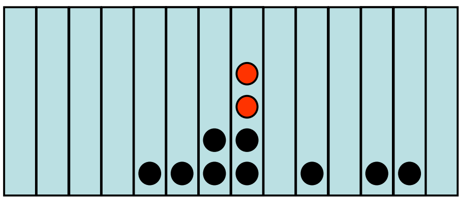
In the following column, that is 1000 (i.e. 8), there are 2 balls. These fall down, leaving zero and producing a remainder in the following column:

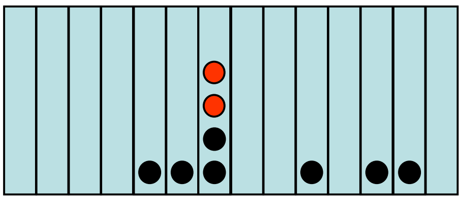


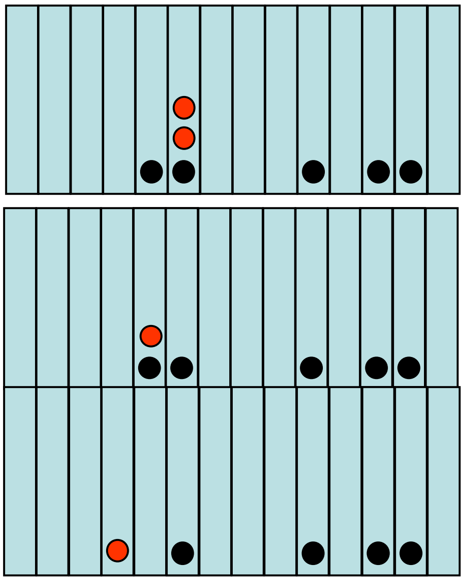
In the 10000 column there are now 3 balls, of which 2 will fall down, producing a remainder. One ball is left over.



This continues step by step until we are left with columns containing either 0 or 1 ball(s). This is the final answer.







1 0 1 0 0 0 1 0 1 1 0

So, the answer of this multiplication is indeed 10100010110!

ATTEMPTS TO BUILD THE MACHINE

This brief manuscript goes beyond these principles and hints at how Leibniz envisaged actually building this machine.

The first box is a repository of balls punched with a line of holes, which when opened or closed to represent 1 or 0, can be used to set the multiplicand. The number of times the box is moved corresponds to the number of powers of 2 in the multiplier and, with each power, a system of two-teeth wheels is used to move a ball to the hole. Here there is a clear analogy with Leibniz's decimal machine.

However, the subsequent lines are less explicit as to how to calculate the remainder. In fact, a "gate" mechanism is needed in order for a ball in one groove to move into the next one along, but only when it is accompanied by another ball. In addition, this second ball has to fall into the box in the underpart of the machine. As Leibniz had come up with far more complex mechanisms for his decimal machine, he would no doubt have resolved this issue had he pursued his reflections on the construction of this binary machine. However, the text itself does not provide any clear indication of how he envisaged building this mechanism.

Despite this lack of precision in the source, machines very similar to what Leibniz's might have looked like have been built on at least two occasions.

Ludolf von Mackensen designed a wooden machine that was built in 1971 by the Deutsches Museum in Munich. An analysis and photos of the machine are available in a thesis by Kasimierz Trzesicki.[13]

Gerhard Weber designed another in transparent material in 2000. A photo is displayed below, while a description is included in an article by Erwin Stein.[14]

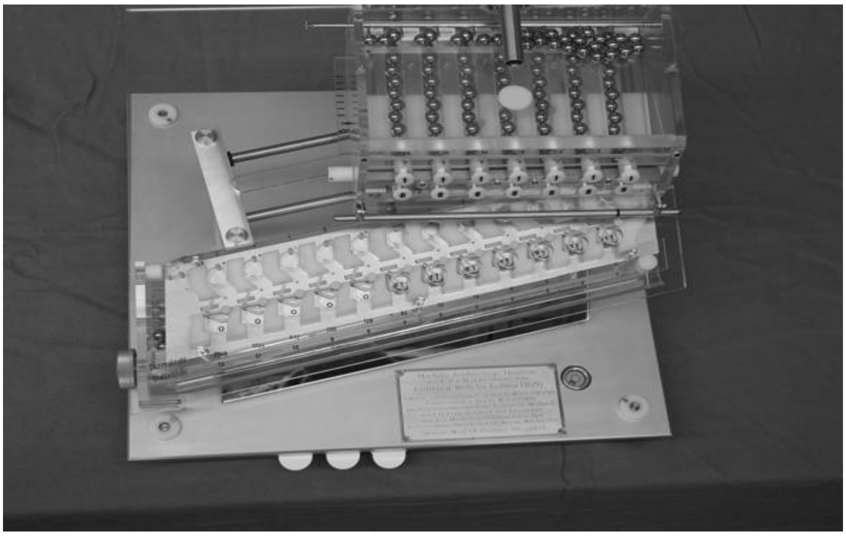


***Figure 5*: Model of a binary machine inspired by Leibniz's text and built in 2003–2004 by E. Stein & G. Weber** *(Institute of Mechanics and Computational Mechanics, Leibniz Universität Hannover)*

While not denying the significant technical extrapolations that were needed to build this machine, it is clear that as early as 1679 Leibniz had recognised the usefulness of building a binary calculating machine and had sketched out a design that has proven operational. He did not take up this idea in his paper published by the Paris Academy of Sciences in 1703, although it does extend the reflections set out in the manuscript of 1679 and contains a description of binary notation as well of the four operations. While clearly unfinished, the treatise *De Progressione Dyadica*, dating from 15 March 1679, is a founding scientific text.

■

(December 2010)

(Translated by Helen Tomlinson, published December 2015)

13. http://logika.uwb.edu.pl/KT/Leibnizjanskie%20inspiracje%20informatyki.pdf (in Polish).
14. http://www.ikb.poznan.pl/fcee/2006.07/full/fcee_2006-07_319-332_calculemus.pdf (*Foundations of Civil and Environmental Engineering*, no. 7, August 2006).