The Book First of Descartes's *Geometry*

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Geometry is the third and last essay in the famous *Discourse on the Method* published by René Descartes in Leiden in 1637. It is the only work of mathematics that he published, but it also the most important, because it had global repercussions, leading in particular to the numericalisation of physics and, from there, to the mastery of the rational universe we now enjoy.

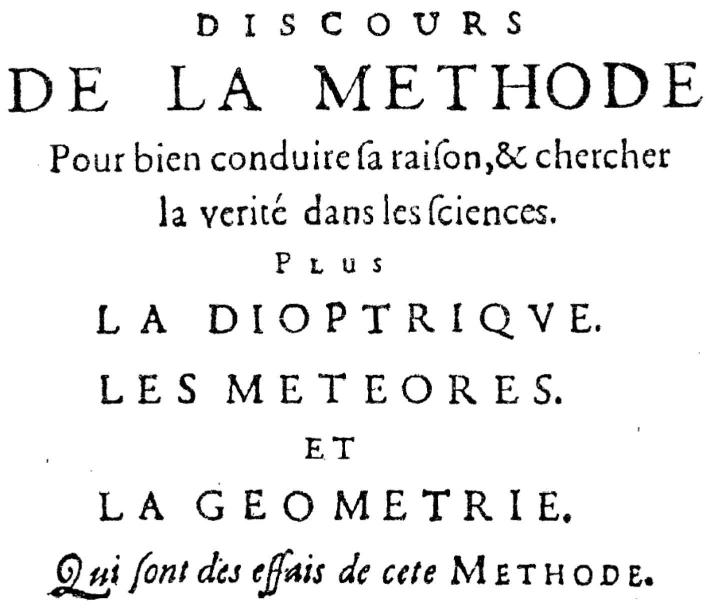
DISCOURS
DE LA METHODE
Pour bien conduire sa raison, & chercher
la verité dans les sciences.
PLUS
LA DIOPTRIQVE.
LES METEORES.
ET
LA GEOMETRIE.
Qui sont des essais de cete METHODE.

***Figure 1*: Discourse on the Method *(1637)*.** *The "Discourse" proper is followed by three "Essays on this Method". Curiously, the annotated edition of the works of Descartes, published by the philosopher Victor Cousin in 1826, separated the initial "Discourse" from the three essays, which are nonetheless an integral part of it (in the Cousin edition, one is in Volume 1 of the complete works, and the others are in Volume V).*

Rightly renowned as the first public intimation of the birth of analytical geometry, this 117-page treatise actually had the hidden aim of providing a general method for the resolution of algebraic equations, that is to say the form $P(x) = 0$, where P is a random polynomial and x the unknown (and by definition real). It is a difficult read; indeed, the author himself stated in a special "notice" at the beginning of the text:

> *Iusques icy i'ay tasché de me rendre intelligible a tout le monde, mais pour ce traité ie crains, qu'il ne pourra estre leu que par ceux, qui sçauent desia ce qui est dans les livres de Geometrie*.

> *Up to this point I have tried to make myself intelligible to everyone. However, for this treatise, I fear that only those who already know what is in geometry books will be able to read it.*[1]

Luckily, this applies to any averagely educated modern reader. We provide here, *in extenso* and in facsimile form, the 18 pages of the original text of the Book First of this work. The Book First is entitled: "Of Problems That Can Be Constructed Using Only Circles and Straight Lines", i.e. with a ruler and compass. Descartes sets out the fundamental elements on which to base the general algorithm for the resolution of equations: the fundamental rules needed to geometrically translate the basic operations (addition, multiplication and even root extraction) are established before the techniques of Cartesian (analytical) geometry are introduced proper, via a description and initial treatment of Pappus's problem, which seems to have been the initial trigger for Descartes's approach.

The first sentence is essential to the understanding of the treatise as a whole:

> *All geometrical problems may be easily reduced to such terms that afterwards one only needs to know the lengths of certain straight lines in order to construct them.*[2]

What this means, in modern language, is that **any geometrical problem can be resolved numerically**.

Descartes applies this sentence to the search for a geometric construction of lines (segments) whose lengths are equal to the roots of a given algebraic equation one wishes to solve, starting – naturally – with those handed down from Antiquity.

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1. --Trans. Descartes's original French and the English translation below are provided in Desmond M. Clark, *Descartes: A Biography*, Cambridge: Cambridge University Press, 2006 p. 151.
2. --Trans. English translation provided in A. Rupert Hall, *From Galileo to Newton*, New York: Dover Publications, 1981, p. 92.

GEOMETRICALLY SOLVING FIRST- AND SECOND-DEGREE EQUATIONS

Descartes first gives the elementary geometric constructions for the product and the quotient of two numbers, as well as the square root of a number on the basis of particular segments, which can be obtained using a ruler and compass. Even on these very simple subjects, he shows himself to be innovative compared to his predecessors, including Viète.

The product and quotient are thus presented as simple applications of what we call Thales' theorem: one need only read in his figure (Figure 1) an $a/b = c/d$ type equality of relations to understand that making one of these numbers equal to 1 (d for example) means we can deduce other relations, such as $a = b.c$ and $c = a/b$. This now gives us a geometric solution to the most general first-degree equation, but also, among other things, a construction of the values of a polynomial or homographic function, etc.

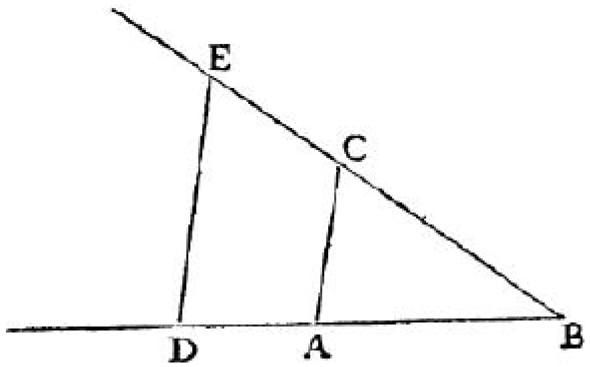

Figure 2: *"For example, let AB be taken as unity, and let it be required to multiply BD by BC. I have only to join the points A and C, and draw DE parallel to CA; then BE is the product of BD and BC"*[3] *(application of Thales' theorem BC/BA = BE/BD).*

It should be noted in passing that his technique of taking a length as a unit, which is now so commonplace, was at the time an innovation of unheard-of abstraction!

The construction of the square root, on this same page, is exactly the same as today: the notations for his figure give us the equality $GI^2 = GH$.

3. --Trans. Unless otherwise stated, the English translations of Descartes's text are taken from *The Geometry of René Descartes*, trans. David Eugene Smith and Marcia L. Latham, New York: Dover Publications, 1954, available at: http://djm.cc/library/Geometry_of_Rene_Descartes_rev2.pdf. This citation: *The Geometry of René Descartes*, p. 5.

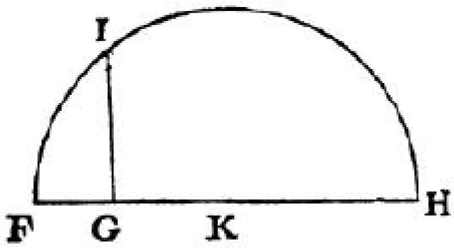

***Figure 3**: "If the square root of GH is desired, I add, along the same straight line, FG equal to unity; then, bisecting FH at K, I describe the circle FIH about K as a center, and draw from H a perpendicular and extend it to I, and GI is the required root." This can be easily verified, by setting down GH = a and seeking its square root (FH = 1+ a ("FG is the unity"), KI = KF = (1+a)/2 (K is the centre of the circle)), then writing $KI^2 = KG^2 + GI^2$, on a $GI^2 = [(1+a)/2]^2 - [(1+a)/2 - 1]^2 = a$; GI is the root we are looking for.*

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Descartes then describes the precepts that need to be followed in his geometry, in a crucial passage (p. 289–302) in which he explains that it is necessary to "name" the different geometric magnitudes of a figure, class them into knowns and unknowns, put them into the form of an equation and then solve these equations.

> *If, then, we wish to solve any problem, we first suppose the solution already effected, and give names to all the lines that seem needful for its construction – to those that are unknown as well as those that are known. Then, making no distinction between known and unknown lines, we must unravel the difficulty … until we find it possible to express a single quantity in two ways. This will constitute an equation, since the terms of one of these two expressions are together equal to the terms of the other.*
>
> *We must find as many such equations as there are supposed to be unknown lines; but if, after considering everything involved, so many cannot be found, it is evident that the question is not entirely determined.*[4]

Let's pause for a moment over the equalities at the top of page 300:[5] the special Cartesian sign  , made up of two lower case *o*'s stuck together, with half the type missing from the first letter – a victim of the printer's chisel – is equivalent to the modern = sign.

Descartes immediately applies the constructions and rules he has introduced to the problems relating to second-degree equations – something he does far better than Euclid, who had dealt with the same subject but in a much more heavy-handed and confused way.

4. --Trans. See *The Geometry of René Descartes*, *op. cit*, p. 6 & 9.
5. --Trans. See *The Geometry of René Descartes*, *op. cit*, p. 8.

This is clear in the figures on pages 302 and 303. The first applies to the case in which the second-degree equation has only one, necessarily positive root, (in this text Descartes deliberately leaves out negative numbers). This root is either the length *MO* or the length *MP*, depending on whether the sum of the two roots is positive or negative. On the second figure, the lengths *MQ* and *MR* are the two – positive – lengths of an equation. Each time, the justifications for these facts are found in the simple notion of power with respect to a circle.

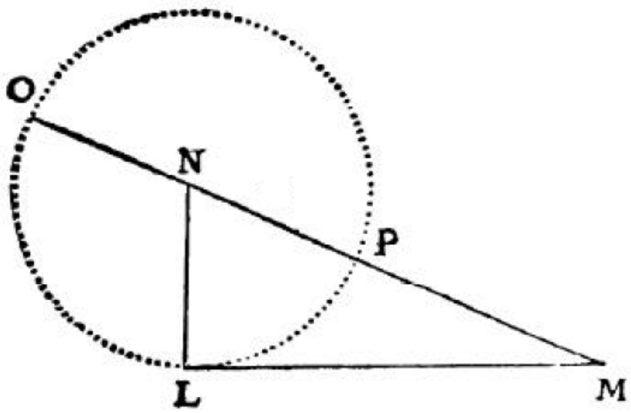


***Figure 3**: "For example, if I have $z^2 = az + b^2$, I construct a right triangle NLM with one side LM, equal to b, the square root of the known quantity b^2, and the other side, LN, equal to ½a, that is, to half the other known quantity … Then prolonging MN, the hypotenuse of this triangle, to O, so that NO is equal to NL, the whole line OM is the required line z. This is expressed in the following way: $z = \frac{1}{2}a + \sqrt{\frac{1}{4}aa + b^2}$." This is indeed the discriminant of the second-degree equation $z^2 = az + b^2$, which we can easily check using the value of z given by Descartes.*

PAPPUS' PROBLEM: THE BIRTH OF ANALYTICAL GEOMETRY

The end of the Book First introduces Pappus' problem, which we have already mentioned. Though surprising to a contemporary reader, this part is nonetheless essential for the two reasons we have already noted: the legitimisation of analytical geometry through the demonstration of a *new* solution to an *old* problem, and the development of a means to provide ever more complex curves on demand – because they were defined by equations of ever higher degree – yet nevertheless introduced on the basis of a geometric preoccupation. This justification seems incongruous today but was necessary at the time, when algebra was still considered to be of minor status (almost a form of entertainment for Arab scholars around the year 1000) compared to the queen of science: geometry.

It is here, almost incidentally one might say (p. 310), that *x* and *y*, the abscissa and the ordinate, make their appearance. These crop up again, for example on page 321 (which is not reproduced here), and more generally throughout the treatise:

> *Call the segment of the line AB between A and B,* x*, and call BC,* y*. Produce all the other given lines to meet these two (also produced if necessary) provided none is parallel to either of the principal lines.*

As essential as these "Cartesian coordinates" appear to us today, for Descartes they are mere tools, just like his method for indeterminate coefficients (p. 347 of Book Second), or his "Cartesian parabola", which is necessary to construct a sixth-degree equation using his method (an innovation of Book Third).

The idea that the invention of this Cartesian geometry can be summed up by the invention of the coordinate axes – however extraordinary that may have been, and Descartes was well aware of it – is therefore a serious mistake; it confuses the means with the end. The author, at least in this initiatory passage, simply considers the *x* and *y* coordinates, fixing a variable point, as two special lengths, which he then makes a point of using to determine all the other geometric elements of the figure. In a way, this concern to classify, or even to automate, brings us back to the *Discourse on the Method*, although it is not explicitly referred to here.

A modern writer would obviously spend less time on Pappus' problem itself, and indeed a non-specialist reader can skip Descartes's painstaking examination, despite its central importance in the development of the Cartesian revolution. In any event, a modern mathematician would no doubt forgo reproducing the Latin translation of the Greek original, at least *in extenso*.

Descartes's presentation of Pappus' problem

Pappus of Alexandria lived in the late third and early fourth century CE. His *magnum opus* is the *Synagoge* or *Collection*, a compilation of eight books which was probably written around 320. Occasionally enriched by original contributions, it is only partially extant (Book I and the first part of Book II are missing), yet remains a precious, often unique, document on the work of earlier mathematicians such as Aristaeus, Euclid, Archimedes and Apollonius, some of whose writings are partly or entirely lost.

Descartes notes in passing: "I cite the Latin version rather than the Greek text, so that everyone may understand." This is Commandino's famous Latin edition (*Pappi Alexandrini Mathematicae Collectiones,* Pisa, 1588), which Descartes reproduces with a few omissions. Here is a literal translation of the essential part; Descartes himself provides a partial French summary (p. 306):

The problem of the locus related to three or four lines, about which he (Apollonius) boasts so proudly, giving no credit to the writer who has preceded him, is of this nature: if three straight lines are given in position, and if straight lines are drawn from one and the same point, making given angles with the three given lines; and if there be given the ratio of the rectangle contained by two of the lines so drawn to the square of the other, the point lies on a solid locus given in position, namely, one of the three conic sections.

In modern language, a Pappus problem "in three or four lines" may be stated as follows. These three or four lines are considered to be "position-given", i.e. known by certain means. For example, we know a couple of distinct points, or a point and the direction, for each of the lines. Given a relation ("proportion" or "reason") ρ, we wish to find the points of a plane such as the "rectangle": the product of the distances from this point to the first two lines ought to be equal to the product by ρ of the square of the distance[6] to the third (or to the product by ρ of the product of the last two, if there are four lines in total).

In modern notation, the meaning of this problem can be expressed as follows: find the points demonstrating a relation of the type $D(1).D(2) = \rho.D(3)$[7] or $D(1).D(2) = \rho.D(3).D(4)$, as the case may be, where $D(i)$ represents the distance to the line number i.

A modern mathematician could, for example, start by establishing a lemma like this: given a frame xOy, a line Δ and an angle φ, the distance CC' – which separates a variable point C in x and y coordinates from the point C' of Δ, such that CC' and this line together form the angle φ – is expressed by the absolute value $|ax+by+c|$ of an expression of the type $ax+by+c$, where a, b and c are independent of the position of the moving point C. (This lemma is a simple corollary of the theorem that the equation of a line is a first-degree polynomial in x and y.)

6. Pappus does talk in terms of distance, which would entail tracing and measuring a segment that is perpendicular to the line in question. Instead he refers to a "straight line" (here this means a *segment*) that makes a given angle with it, though not necessarily a right angle. This generalization – a kind of oblique distance – is insignificant as it simply amounts to modifying the value of ρ.

7. See footnote 6.

Descartes does not say any of this – but he applies it. It becomes immediately obvious that a Pappus locus "made of four lines" is a conic section that has an equation in the form

$$(ax+by+c).(dx+ey+f) = \rho.(gx+hy+i).(jx+ky+l)$$

since all of the points *C*, such as four distances of the type *CC'* above, are linked in such a way that the product of the first two remains proportional to that of the two others.

Descartes spends a great deal of time expanding on this example – before returning to Book Second with a dazzling first example of Cartesian calculation to prove what he has said, showing how to substitute manipulations of numbers with purely geometric, Euclidean-type reasoning.

IN DESCARTES'S WORDS: RANDOM-DEGREE CURVES

Extending Pappus' initial problem, Descartes introduces the locus related to $2n$ lines, and their variants related to $2n-1$ lines:

> *And thus this problem can be extended to any number of lines. Then, since there are always an infinite number of points which can fulfil the requirement, here it is also necessary to know and to trace the line in which they are to be found.*

In the same way, these loci are recognised as being what we now call *algebraic curves* of the nth degree, defined by equations in the form $F(x,y) = 0$, where F is a polynomial of nth degree. (The fact that any curve of this kind admits a Pappus-type definition is a particularly thorny problem, and one with a negative solution. Descartes does not raise this point here.) Pappus' problem can indeed be extended to a family of $2n$ lines, in the form $D(1).D(2)\ldots D(n) = \rho.D(n+1).D(n+2)\ldots D(2n)$, or to a family of $2n-1$ lines, this time in the form $D(1).D(2)\ldots D(n) = \rho.D(n+1).D(n+2)\ldots D(2n-1)$ for $n \geq 3$ (the case $n = 2$ held special significance for Descartes, as we just saw).

The positions of the solutions to such a Pappus problem produce a curve (or a line, or *geometric line*), which is formed by an infinite number of elements (or points). It is impossible to find all of these (because one can only ever trace a finite number of points). As what Descartes is asking is to "know" (or to "find" or "trace") such a curve, that means he considered the problem to be solved when it was possible to find an infinite number of the points where it passes, i.e. to determine (by solving algebraic equations) all the points of the locus situated on a random line for which only the direction is fixed. (The lines of a plane with a set direction do indeed form a family, the meeting point of which is the plane as a whole: knowing all the points situated on these lines amounts to knowing all the points forming the curve.) For any *y*-parameter value, this technically amounts to being able to solve, for the unknown *x*, the equation $F(x,y) = 0$, which gives the abscissas of the points of the *y*-ordinate curve, using geometric constructions like those Descartes had given earlier and later expands on in Book Third.

An example of a Pappus problem with four lines (n = 2)

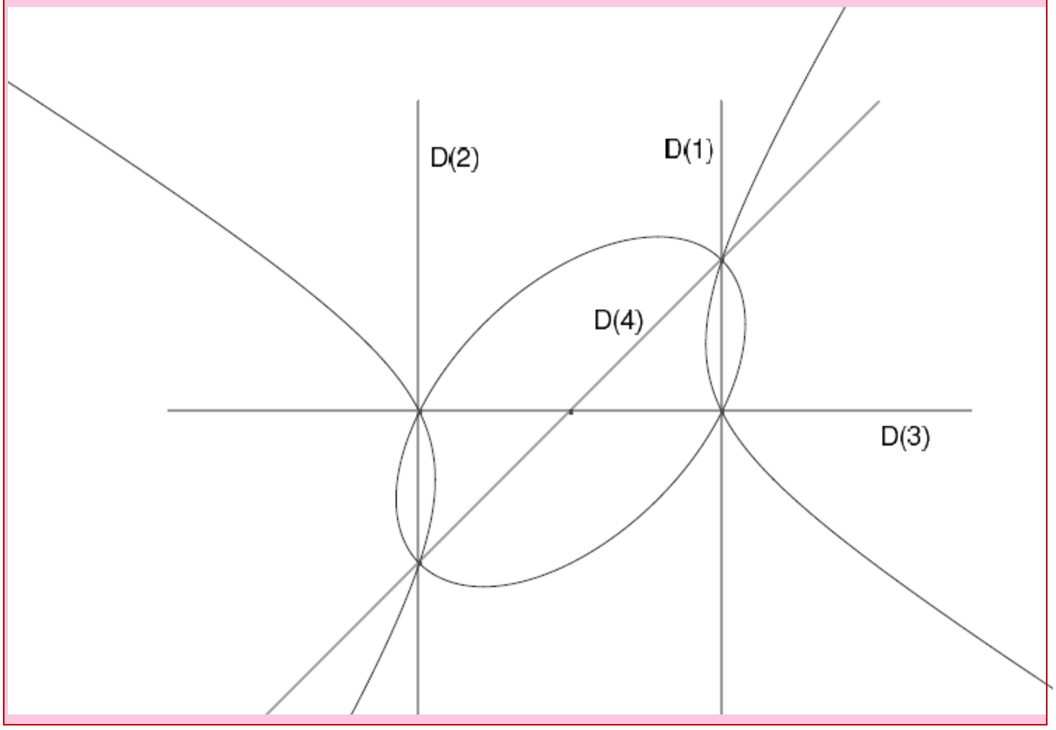
Let there be four lines D_1, D_2, D_3, D_4 with the respective equations x=a, x=-a, y=0 and y=x (see figure below). If we note D_i, the distance of a point M from the Di line, we can try to find the locus of the M points, such as $D_1.D_2=\sqrt{2}.D_3.D_4$

We have $D_1 = |x\text{-}a|$, $D_2 = |x+a|$, $D_3 = |y|$ and $D_4 = 1/\sqrt{2}.|x\text{-}y|$ (the calculation for this last distance is quite simple: the point *x,y* is projected orthogonally onto the line $x = y$, following a line with a slope of -1).

The locus of the M points is therefore given by $|x^2\text{-}a^2|=|y(y\text{-}x)|$, i.e. $x^2 - a^2 = y^2 - yx$, or $x^2 - a^2 = xy - y^2$

There are two conic solutions, each with a centre O, and each passing through the four points (a,0), (a,a), (-a,0), (-a,-a):

- the equilateral hyperbola with the equation $x^2+xy\text{-}y^2=a^2$, which also passes through the points (2a,-a) and (2a,3a)
- the ellipse of the equation $x^2\text{-}xy+y^2=a^2$, which also passes through the points (0,a) and (0,-a).



THE INVENTION OF ANALYTICAL GEOMETRY

On page 307, Descartes affirms, at this point without proof (see Book Second), that if the Pappus problem is in three, four or five lines (with the exception of the case of five parallel lines), it is possible to construct the locus point by point using the usual means, that is to say the ruler and compass (the problem is therefore *plane*).

> *First, I discovered that if the question be proposed for only three, four, or five lines, the required points can be found by elementary geometry, that is, by the use of the ruler and compasses only.*

When more lines are involved, it is necessary to turn to more elaborate curves. Indeed, it is in order to arrange these that he introduces the coordinates that now bear his name.

On page 309 and the pages that follow he shows how to express a Pappus problem in four lines in the form of an equation (from which the equivalent problem in three lines can be easily deduced). This is, on the face of it, a specific case (ρ relation equal to 1), but "to take another given proportion" would change almost nothing at all.

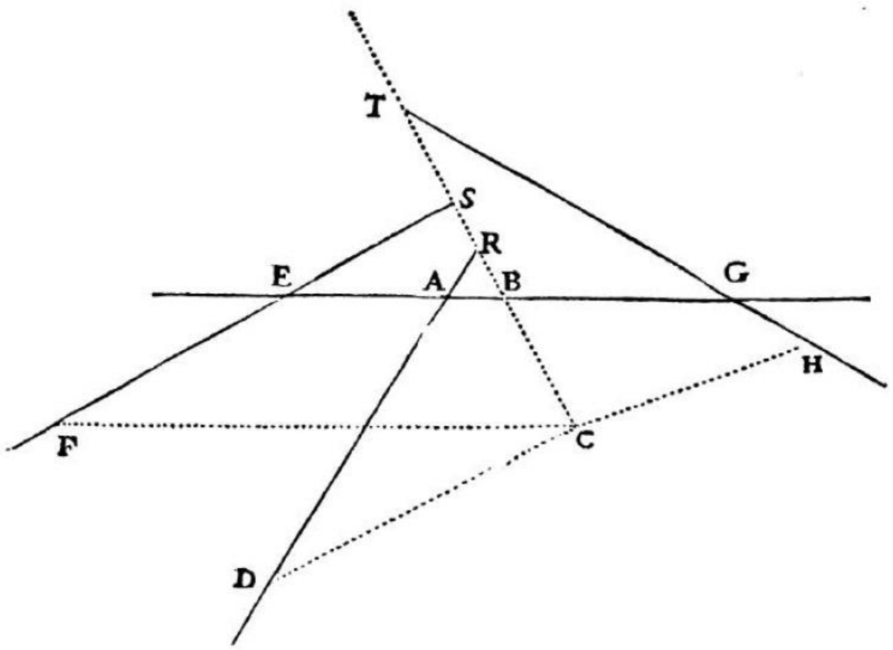


Figure 4: *Descartes's presentation of a Pappus problem in four lines (AB, AD, EF, GH).*

Descartes starts by taking four straight lines, $D(1) = AB$, $D(2) = SEF$, $D(3) = RAD$ and $D(4) = TGH$, which, along with the points *AEG*, are the only fixed elements in the configuration. The lines *CBRST*, *CF*, *CD* and *CH* are variable (although they have fixed directions), while the points *B, F, D, H, S, R* and *T* respectively describe parts of the lines *D(*1), *D(*2), *D(*3), *D(4*), *D(*2), *D(*3) and *D(*4), with point C serving to describe the geometrically studied locus.

To produce the equation, he reduces the calculations of all the lengths he needs to just two, namely $x = AB$ and $y = BC$. In comparison with our usual coordinates, it should be noted that *A* is the start of the frame, and *D(1) = AB* the axis of the abscissas. The axis of the ordinates is not traced onto the figures on pages 309 and 311, but it would be easy to draw it as being the parallel of *CBRST* (with a known direction), issuing from point *A*.

For the first time in history, our geometrician and the mathematical community as a whole had access to objects of study that could be however complex they wished to make them, as well as tools to produce a fundamental algorithm capable of solving algebraic equations. These nevertheless had an "old-style" definition, thereby ensuring their continuity with the masters of antiquity, who, aside from a few special cases invented for the sake of the cause (trisectional curves, for example), had a rather limited repertoire, essentially limited to conic sections.

This was a considerable quantitative and qualitative leap. For example, thanks to these highly original tools, fifth- and sixth-degree equations could be solved *in fine* for the first time. It is thus a visibly and justifiably proud author who, in the conclusion to his book, announces that he must now establish the properties of the new beings his discovery has engendered, something he proceeds to do in Book Second and Book Third of his *Geometry*.

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(November 2009)

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