Leibniz's differential calculus applied to the catenary

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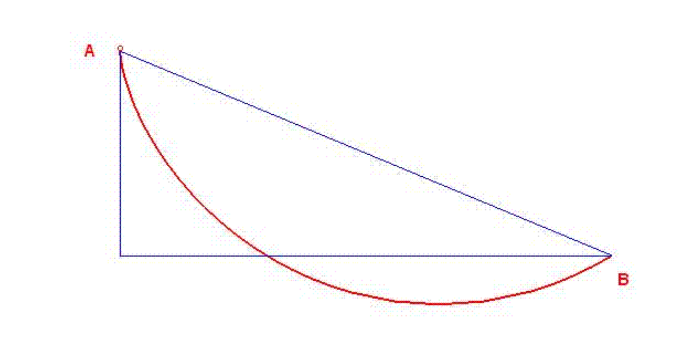
Study of the catenary was a response to a challenge laid down by Jacques Bernoulli, and which was successfully met by Leibniz as well as by Jean Bernoulli and Huygens: to find the curve described by a piece of strung suspended from its two ends. Stimulated by the success of this initial research, Jean Bernoulli put forward and solved similar problems: the form taken by a horizontal blade immobilised on one side, with a weight attached to the other; the form taken by a piece of linen filled with liqueur; the curve of a sail.

Challenges among scholars

In the 17th century, it was customary for scholars to set each other challenges in alternate issues of journals. Leibniz, for example, challenged the Cartesians – as part of the controversy surrounding the laws of collision, known as the "querelle des forces vives" (vis viva controversy) – to find the curve down which a body falls at constant vertical velocity (isochrone curve). Put forward in the *Nouvelle République des Lettres* of September 1687, this problem received Huygens' solution in October of the same year, while Leibniz's appeared in 1689 in the journal he had created, the *Acta Eruditorum*. Another famous example is that of the brachistochrone, or the "curve of the fastest descent", which, thanks to the new differential calculus, proved to be not a circle, as Galileo had believed, but a cycloid: the challenge had been set by Jean Bernoulli in the *Acta Eruditorum* in June 1696, and was resolved by Leibniz in May 1697.
It was also in this journal, in May 1690, that Jacques Bernoulli suggested that Leibniz examine if his new calculus could solve the problem of the catenary. Leibniz did indeed solve this problem, though he did not publish the solution – to give other mathematicians time to try their hand at it – until 1690. Huygens, Jacques and Jean Bernoulli successfully solved the problem in the imparted time.[1]

1. As Marc Parmentier indicates in his annotations accompanying Leibniz's text (Éditions Vrin): "The historical prestige of the question and the competition solemnly installed by Leibniz had put the catenary in everyone's minds and in the years to come paved its way to the career of the new cycloid. However, the particularity of the



<u>Figure 1:</u>* Brachistochrone curve. *The arc of the cycloid AB (in red) is the curve through which a weight leaving from A will arrive at B in the shortest amount of time.*

It should be noted that some of the curves studied by Leibniz were already known, and that many of their properties had already been established through purely geometric means or pre-differential methods. The cycloid, for example, was one such case, but differential calculus could pride itself on having considerably simplified the proofs of the properties already established. In contrast, the catenary was a curve ***discovered*** thanks to the new calculus, which could therefore boast of having genuinely advanced *the art of inventing*, of which Leibniz so fond. Moreover, the author implies that he was the first to have noticed and used the link between this curve and logarithms, a link that makes it possible to "construct" the latter simply with a suspended piece of string.

> *Indeed, I observed that the fecundity of this curve is equalled only by the ease by which it is realised, which places it ahead of all the Transcendents* [transcendental curves]. *In fact, we can obtain and trace it with little difficulty, through a physical type of construction, by hanging a piece of string or better still a catenary (of invariable length). And once, thanks to this, we have its outline, we can bring to light … all the Logarithms we may wish …*

We will untangle Leibniz's difficult text by first giving the differential equation of the catenary, which, though it forms the basis of the text, is absent from it. From this, we will deduce point by point the construction of the catenary, and, inversely, the construction of logarithms from a "suspended piece of string". Finally, we will deduce some of the properties of the catenary, stated by the author without demonstration.

catenary, which no doubt contributed to its prestige, is that it is not a mechanical curve in Descartes' sense, but a static curve. As such, it outclasses the cycloid in the scale of the transcendent curves to the extent that it cannot be constructed through a composition of movements."

***Figure 2**: The catenary is the form taken by a piece of string, or a chain, suspended from its two ends at equal height.*

THE EQUATION OF THE CATENARY

We can retrieve this equation thanks to a passage from Montucla:[2] "We cannot, we believe, avoid putting geometrician readers some way down the path to the solution of this curious and difficult problem. To do so we will borrow the subtle analysis given by Jean Bernoulli in his *Lectiones calculi integralis*."

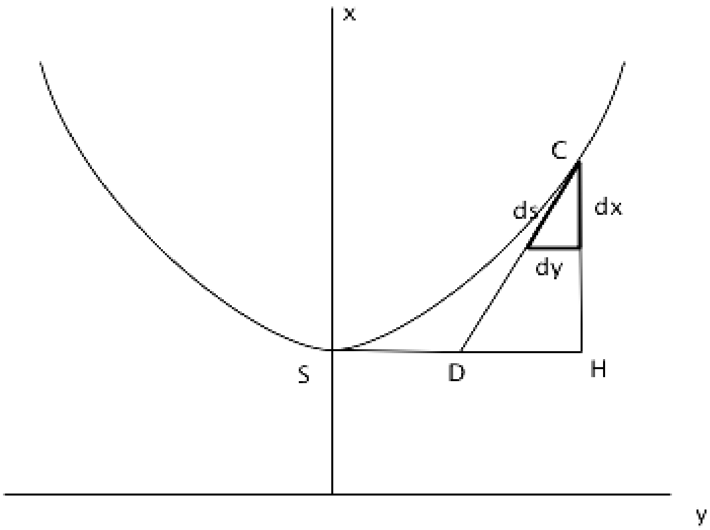


***Figure 3:** The equation of the catenary, established by Jean Bernoulli, stems from the similarity between the triangle CDH and the characteristic triangle (dx, dy, ds). Note the inversion of the axes compared with contemporary conventions. The constant* a *below is the abscissa of the point S (on the vertical axis).*

Jean Bernoulli's reasoning is based on a property of statics, according to which the weight of section *SC* of the string (proportional to the length *s* of this

2. *Histoire des mathématiques*, Tome II, p. 468.

section) is to the constant tension (noted *a*) of the string in *S* as *CH* is to *DH*, where *CD* is the tangent to the string in *C* (Fig. 2). By returning to Montucla's (slightly modernised) text, we can establish this property as follows.

The equilibrium of a portion *SC* of the string (Fig. 2) results from the equilibrium of three forces: the tension *a* of the string in *S*, the tension *T* of the string in *C*, and the weight *s* (by way of a certain choice of unit) of the portion of the string *SC*. The tensions are "powers" (forces) directed by the tangents to the points in question; the tension in *S* is therefore directed horizontally, and the tension in *C* is directed following *CD*.[3]

Taking the components of the three forces brought to point *D*, we therefore have:

- horizontally: *a* and $T\times$ cos *CDH*, therefore $a = T\times \cos CDH$
- vertically: the weight *s* and $T\times$ sin *CDH*, therefore $s = T\times \sin CDH$

We deduce that $\frac{s}{a} = \tan CDH = \frac{CH}{DH}$

Considering the characteristic triangle in *C*, we have $\frac{CH}{DH} = \frac{dx}{dy}$. The differential calculus of the catenary is therefore $\frac{s}{a} = \frac{dx}{dy}$, "an equation which, treated adroitly", Montucla tells us, "will be reduced to $dy = \frac{adx}{\sqrt{x^2 - a^2}}$".

This "adroit" treatment, which Montucla leaves up to the geometrician reader, should look like this: as $ds^2 = dx^2 + dy^2$, $\frac{s^2}{a^2} = \frac{dx^2}{dy^2} = \frac{dx^2}{ds^2 - dx^2}$, therefore $\frac{dx}{ds} = \frac{s}{\sqrt{a^2 + s^2}}$ and, consequently, by integrating, $x = \sqrt{a^2 + s^2}$, hence $s = \sqrt{x^2 - a^2}$. By carrying this value of *s* to the initial equation $\frac{s}{a} = \frac{dx}{dy}$, we obtain the equation sought: $\mathbf{dy = \frac{adx}{\sqrt{x^2 - a^2}}}$ **(1)**

3. Importantly, the tension *a* in *S* is independent of the choice of point *C*.

Note that we obtain the equation[4] of only part of the curve, the increasing part, because the coordinates are pairs of *positive* numbers. The difficulties that such a restriction might raise, at a time when negative numbers were not yet recognised as legitimate, can possibly be compensated here by symmetrical considerations: we have the same curve "on the other side".

CONSTRUCTION OF THE CATENARY

> *Here is a* geometric construction of the curve, *without the aid of any string or any catenary, and without the supposition of quadratures, a construction which, to my mind, must be judged the most perfect that can be obtained for the transcendents, and the one most consistent with the analysis.*

The text that follows this passage is difficult for the contemporary reader, all the more so since Leibniz, devilishly elliptical as he often is – and who has not even given the equation of the catenary – does not explain the link between the latter and a "logarithmic curve", which is nevertheless the essential link in his text. Here is a differentiation possible at this time.[5] We have $\frac{dy}{a} = \frac{dx}{s}$, and we have seen that $s = \sqrt{x^2 - a^2}$, therefore $ds = \frac{xdx}{s}$. As a consequence:[6]

$$\frac{dy}{a} = \frac{dx}{s} = \frac{ds}{x} = \frac{d(x+s)}{x+s}$$

Hence the solution is $y = a\log(x+s)$; the constant is determined by setting down the condition $y = 0$ for $x = a$ (Fig. 2), which gives:

$y = a\log(\frac{x+s}{a})$ (because for y = o, we have s = length of the arc = 0)

We will take $a = 1$ as Leibniz suggests, hence ultimately:

$$\mathbf{y = \log(x+s) \quad (2)}$$

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4. In contemporary notations, the solution of this equation is $y = a \times \ln\left(\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 - 1}\right)$. The reader will be able to verify that $x = \frac{a}{2}\left(e^{y/a} + e^{-y/a}\right)$, which is indeed the equation of a catenary, with Leibniz's customary inversion of axes.
5. Following an idea by Dominique Bénard of the Mans Irem (Institut de recherche sur l'enseignement des mathématiques).
6. Here we are applying a property linked to fractions, if a/b = c/d then this relation is equal to (a+c)/(b+d).

We observed above that equation **(1)** concerns only half of the curve. The other half, which is symmetrical to the first in relation to the vertical axis, will have the following equation:

$$y = -\log(x+s) = \log\left(\frac{1}{x+s}\right) = \log(x-s) \quad \textbf{(3)}$$

since $(x-s)(x+s) = x^2 - s^2 = a^2 = 1$.

The principle of construction of the catenary of equation **(2)** and **(3)** is therefore as follows: once the logarithmic curve has been constructed, let us take two points of this curve of abscissas *y* and *–y*. The corresponding ordinates give us *x+ s* and *x – s*. The ordinate *x* of the point of the catenary of abscissa *y* will therefore be the half-sum of the two preceding ordinates.

To untangle the construction of the logarithmic curve as set out by Leibniz, one simply has to bear in mind that this curve joins up points whose abscissas *y* are in arithmetic progression, whereas the corresponding *x* ordinates are in geometric progression. To construct a logarithm with a given base $k > 1$, thus a curve of the type ($x = \log_k y$, y), he essentially proceeds as follows: the first points are (1 ; *k*), (-1 ; 1/*k*), (0 ; 1). To then have the ordinate of the point of abscissa ½, we construct the "mean proportional"[7] between 1 and 1/k; similarly, the abscissa of the point of ordinate -1/2 is the "mean proportional" between 1 and 1/k, i.e. $\frac{1}{\sqrt{k}}$. We have therefore constructed, in abscissa, an arithmetical sequence with a common difference of ½, and, in ordinates, a geometrical sequence with a common ratio of $\sqrt{k}$.

We can take this further by constructing step by step points (n ; k^n) and ($-n$, k^{-n}), then points ($n+1/2$; $k^{n+1/2}$), where $k^{n+1/2}$ is obtained as the mean proportional of k^n and k^{n+1}. We also continue by bisection: we place the mean proportional of 1 and of $\sqrt{k}$ at the point of abscissa ¼, and do likewise for all the other segments, so as to obtain in abscissa an arithmetical sequence with a common difference of ¼ and in ordinates a geometrical sequence with a common ratio of $k^{1/4}$. In contemporary terms, this is a point-by-point construction of the curve $x = k^y$ or $y = \ln_k x$.

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7. Nowadays, we would tend to say geometric mean: the geometric mean of *a* and of *b* is the square root of the product *ab*, which we can construct with a ruler and compass using two segments of length *a* and *b*. The same applies to the progressive construction of the integral powers of *k* from a unit segment and a segment of length *k*.

INVERSELY, FINDING THE LOGARITHMS OF NUMBERS USING THE CATENARY (FIG. 4)

Inversely, if the Catenary is constructed physically, by suspending a string or a chain, we can thereby establish as many mean proportionals as we wish, and find the logarithms of given numbers, or the numbers of given logarithms.

Transcribed into contemporary language, Leibniz's construction is as follows. As the catenary with the equation $x = \frac{1}{2}(e^{y} + e^{-y})$ is traced by "hanging a piece of string" (Fig. 3), it will therefore pass through the point of coordinates ($y = \ln a$; $x = \frac{1}{2}\left(a + \frac{1}{a}\right)$) for a given $a > 0$. To construct $\ln a$, knowing a, we will therefore take a on the x axis, and we will construct[8] $\frac{1}{2}\left(a + \frac{1}{a}\right)$, which we will transfer onto this same axis; we will also take one of the two elements of its inverse image on the y axis. It is true that this gives two solutions corresponding to $\ln a$ and $\ln(1/a)$, but Leibniz considers both to be positive and therefore equal. Inversely, if we wish to construct a knowing $\ln a$, we will again use the point of the catenary of abscissa $\ln a$. In effect, its ordinate is $\frac{1}{2}\left(a + \frac{1}{a}\right)$, thus $s = \left(a + \frac{1}{a}\right)$ is known; to obtain a, all that remains to do is to construct two segments p and q that are the inverse of one another and of which the sum s is given. As Leibniz says, it's child's play: we can take, for example, the abscissas the points of intersection of the straight line $x + y = s$ and of the hyperbola $xy = 1$.

8. A segment of length a being given, we can in effect geometrically construct a segment of length $1/a$, then the half-sum of those preceding it.

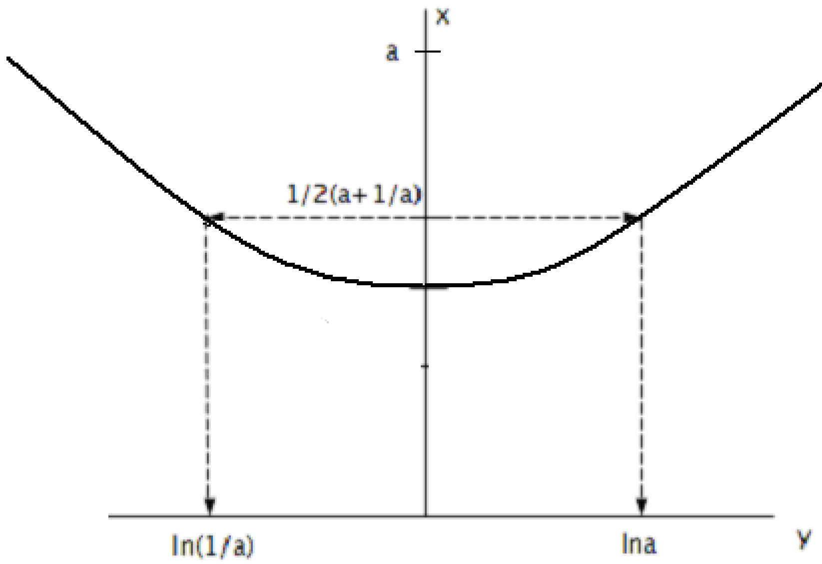

***Figure 4**: Construction of lna from the outline of the catenary (here, for example, we have taken a=2).*

The author then states a whole series of the catenary's properties without any demonstration, indeed without even giving an equation for the curve. We are now going to recover these properties using the instruments of the time, in the manner of the author of a late 17th-century textbook.

TANGENT TO A POINT OF THE CATENARY (FIG. 5)

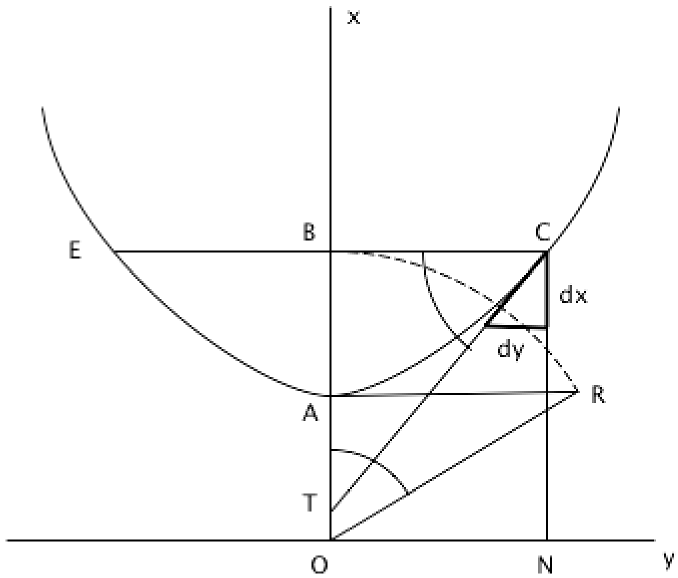

***Figure 5**: **Properties of the catenary**. By placing R on a line parallel to Oy such that OR = OB, the tangent CT is such that the angle BCT is equal to the angle AOR, AR is equal to the arc AC, and the area of the mixtilineal area AONCA is equal to that of the rectangle with sides OA and AR.*

Let the question be to determine the tangent *CT* in *C* to the catenary of vertex *A*, where *OA* = *a*. Let's place point *R* such that *OR* = *OB* is on a line parallel to *Oy* passing through *A*. By taking into account equation **(1)** and the fact that the triangle *CBT* is similar to the characteristic triangle in *C* to the curve, we will have:

$$\frac{CB}{BT} = \frac{dy}{dx} = \frac{a}{\sqrt{x^2 - a^2}} = \frac{OA}{\sqrt{OB^2 - OA^2}}$$

But as *OB* = *OR*, $\sqrt{OB^2 - OA^2} = \sqrt{OR^2 - OA^2} = AR$, hence ultimately $\frac{CB}{BT} = \frac{OA}{AR}$.

The triangles BCT and AOR are therefore similar, thus the angle AOR is equal to the angle BCT, and thereafter the angles BCT and ARO are complementary: Leibniz calls this the "antiparallelism" of the straight lines *CT* and *OR*.

> *Here as a shorthand I name as antiparallels the straight lines OR and TC which, with the parallel lines AR and BC, make the angles ARO and BCT, which are not equal but are nevertheless complementary. The triangles OAR and BCT are therefore similar.*

Hence the construction of the tangent *CT*: place the point *R* as indicated above, then construct[9] the straight line antiparallel to *OR* passing through *C*.

Rectification of a catenary arc (Fig. 5)

Let the question be to find a segment of a straight line with the same length *s* as the arc *AC*.

By establishing the equation of the curve, we have found that $s = \sqrt{x^2 - a^2}$, therefore:

$$s = \sqrt{x^2 - a^2} = \sqrt{OB^2 - OA^2} = AR,$$

as we saw above.

Thus the arc *AC* has the same length as the segment *AR.*

9. This type of construction was easy: once the point R was constructed, one knew the angle *AOR*, which was transferred onto *BC* in *C* to construct the angle *BCT*.

QUADRATURE (FIG. 5)

Let the question be to square the mixtilineal area *AONCA*. Its area is:[10]

$$\int xdy = \int a\frac{xdx}{\sqrt{x^2 - a^2}} = a\sqrt{x^2 - a^2} = OA \times AR$$

It therefore has an area equal to that of the rectangle whose sides are *OA* and *AR*.

As Leibniz says, "we also see that the arcs are proportional to the quadrilineal areas", since the arc of the curve *AC*, whose length is that of the segment *AR*, is proportional to the area *OAxAR* of the area *AONCA*.

CENTRE OF GRAVITY OF A PORTION OF A CURVE (FIG. 5)

Let the question be to find the centre of gravity of the portion of the curve *CAE*, where *E* is symmetrical to *C* with regard to the axis of the abscissas. Through symmetry, the centre of gravity has a zero ordinate. The abscissa of the centre of gravity of a finished system of points of abscissa x_i and masses m_i, and a total mass *M*, is equal by definition to $\frac{1}{M}\sum m_i x_i$. By passing to an infinity of points of a curve of abscissas *x*, and by assuming that the mass is proportional to the length, the total mass will be *s* and the mass of each point will be *ds*: the abscissa of the centre of gravity will therefore be $\frac{1}{s}\int xds$.

$s = \sqrt{x^2 - a^2}$, $ds = \frac{xdx}{\sqrt{x^2 - a^2}}$, and thereafter:

$$\int xds = \int \frac{x^2}{\sqrt{x^2 - a^2}}dx = \int (\sqrt{x^2 - a^2} + \frac{a^2}{\sqrt{x^2 - a^2}})dx = \int sdx + ay$$

according to formula **(1)**.

But $\int sdx = sx - \int xds$ (Fig. 5), hence $\int xds = sx - \int xds + ay$, that is to say $2\int xds = sx + ay$.

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10. Here, as in the following paragraph, we perform the calculation *à la Leibniz*, without specifying the limits. Nowadays, we would write: $\int_0^y xdt = \int_a^x a\frac{udu}{\sqrt{u^2 - a^2}}$ etc.

The abscissa of the centre of gravity is therefore: $\frac{1}{s}\int xds = \frac{1}{2}(x + \frac{ay}{s}) = \frac{1}{2}(OB + \frac{OA \times BC}{AR})$. This last formula corresponds to the construction indicated by Leibniz, since the "proportional fourth"[11] *z* of *AR*, *BC* and *OA* is defined by $\frac{AR}{BC} = \frac{OA}{z}$, hence $z = \frac{OA \times BC}{AR}$

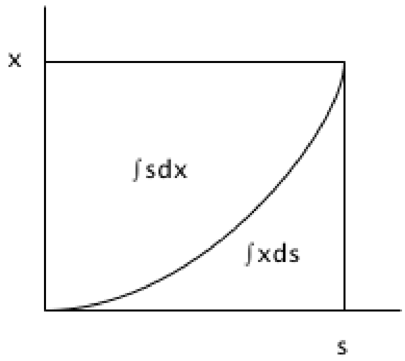

Figure 6: *What we today call the integration by parts formula ∫xds + ∫sdx = sx can be seen directly on the figure. This is the very simple fact that the two "mixtilineal triangles" are "complementary", in Leibniz's own words.*

By way of conclusion: The mathematical posterity of the catenary

As Marc Parmentier underlines: "Let us note last of all that by methodically setting out the remarkable properties of the catenary, Leibniz sketched the plan of study of the curve with admirable clarity: establishment of the tangents, rectification of an arc, calculation of area, centre of gravity of an arc, centre of gravity of the area, surface and volume of the solid of revolution".

Discovered as the solution to a problem of physics, the equation of the catenary underwent a significant purely mathematical development in the form of the *hyperbolic cosine*. The idea, which we owe to Vincenzo Ricatti (1757), and which was developed by Jean-Henri Lambert from 1761 onwards,[12] is that by setting down $chx = \frac{1}{2}(e^x + e^{-x})$ and $shx = \frac{1}{2}(e^x - e^{-x})$, we have $ch^2x - sh^2x = 1$; as a consequence, ($x = cht$; $y = sht$) is a parametric equation

11. I.e. the fourth term entering the equality of quotients that follows.
12. A text of Lambert's on the irrationality of π and the first constructions of hyperbolic trigonometry can be found on BibNum (February 2009).

of the hyperbola $x^2 - y^2 = 1$. With regard to the hyperbola, the hyperbolic cosine *ch* and the hyperbolic sine *sh* therefore play a role analogous to that which the ordinary cosine and sine have with regard to the circle, since (x = cos$t$; y = sin$t$) is a parametric equation of the unit circle.[13]

The catenary (with the equation y = ch x), revealed through the analysis of its physical properties, would thus be at the origin of two important mathematical developments: hyperbolic trigonometry and differential calculus.

■

(March 2009)

(This text was revised and adapted by the author on the basis of the IREM book he coordinated, Textes fondateurs du calcul infinitésimal, *2006, with the kind permission of Éditions Ellipses)*

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13. Especially since, as Euler had established in 1740, we have $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$.