Argand's geometric representation of imaginary numbers

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THE REASONS FOR A CHOICE

Choosing Argand's *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques* (1806) as a founding and representative text about a major advance in mathematics could prove risky – for two reasons.

Firstly, the biographical details about Argand himself remain uncertain. We can restate what has often been written: Jean-Robert Argand was a Swiss mathematician born in Geneva in 1768, who settled in Paris as a bookseller and remained there until his death in 1822. But, in the re-edition of this *Essai* in 1874, published by Gauthier-Villars, J. Houël points out in his preface:

> Nous aurions vivement désiré de pouvoir donner à nos lecteurs quelques renseignements sur la personne de l'auteur de cet important Opuscule. Nous nous sommes adressé, pour en obtenir, au savant le plus versé dans l'histoire scientifique de la Suisse, à M. R. Wolf, à qui l'on doit un *Recueil de Biographies* aussi remarquable par la profonde érudition que par l'attrait du récit. M. Wolf a eu l'obligeance de faire faire aussitôt des recherches à Genève, ville natale d'Argand. Malheureusement les informations qu'il a pu se procurer, par l'intermédiaire de M. le professeur Alfred Gautier, se réduisent à quelques lignes, que nous transcrivons ici :
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> « J'ai bien trouvé l'inscription de la naissance, le 22 juillet 1768, de JEAN-ROBERT ARGAND, fils de Jacques Argand et de Ève Canac. C'est très-probablement l'auteur du Mémoire de Mathématiques en question. D'après ce qui m'a été dit par une personne qui connaissait sa famille, ce monsieur a été longtemps teneur de livres à Paris, et je présume que c'est là qu'il est mort.

[We would have very much liked to provide our readers with a few details on the personage of the author of this important opuscule. In order to obtain these, we turned to the scholar the most well versed in the scientific history of Switzerland, M. R. Wolf, to whom we owe a Recueil de Biographies *as remarkable for its deep erudition as for the appeal of the narrative. M. Wolf was obliging enough to immediately conduct research in Geneva, Argand's home city. Unfortunately, the information he was able to procure, through Professor Alfred Gautier, amounts to a few lines, which we transcribe here:*

"I did indeed find the entry of birth of Jean-Robert Argand, son of Jacques Argand and Éve Canac, dated 22 July 1768. This is most probably the author of the Essay of Mathematics in question. According to what I was told by a person who knew his family, for many years this gentleman was a bookseller in Paris, and I presume it was there that he died."]

These details remain controversial to this day. The only thing we know for sure is stated by Houël in this same text: in 1813, Argand resided at 12, rue de Gentilly in Paris, as is indicated by the handwritten note he appended when he sent a copy of his *Essai* to Joseph-Diez Gergonne.

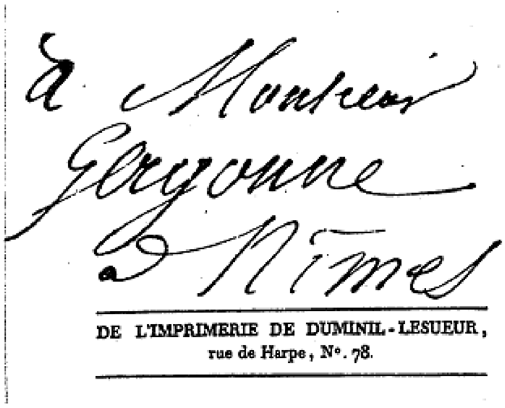

***Figure 1**: Argand's dedication to Gergonne, on the last page of his essay of 1806.*

Moreover, a piece of work completed earlier than Argand's, but which was discovered much later, is considered by many historians of mathematics to be the true founding text of the geometric representation of imaginary numbers. This is the text by the Dane Caspar Wessel (1745–1818), which was published in 1799 in the memoirs of the Royal Danish Academy of Sciences and Letters. This essay, *On the Analytical Representation of Direction*, only really came to light at the end of the 19th century, in a French translation published in Denmark in 1897. There is, however, no mystery surrounding Wessel's biography.

Nevertheless, we are taking the risk of viewing Argand's *Essai* as being of seminal importance in the geometric representation of complex numbers. There are several reasons for this.

It is often difficult to trace one's way back to the true sources of a concept, given that many avenues may have led to its emergence. A founding text is thus considered to be the one that established the concept in the field of knowledge to which it pertains, and even beyond, and which laid down all the characteristics and implications of that concept. This was not true of Wessel's essay, but it was of Argand's.

Furthermore, Argand's *Essai*, when published in simplified and enhanced form in Gergonne's *Annales de mathématiques pures et appliquées* in 1813,[1] provoked numerous reactions and advances based on the new framework it offered to mathematics: it was thus a new point of departure in this science.

Argand himself perceived the new possibilities offered by his geometric representation of imaginary numbers, since the following year, again in the *Annales de Gergonne*, he put forward a demonstration of the fundamental theorem of algebra based on the "directed lines" of his model.[2]

Others who succeeded him attempted other generalisations in the third dimension which presaged the theory of quaternions, such as J. F. Français[3] in his letter on the theory of imaginary numbers, with notes by Gergonne, in 1815 (*Annales de Gergonne*, Tome IV, pp. 222–227).

Argand's work (his *Essai* of 1806, his articles, and those of J. F. Français seven years later in the *Annales de Gergonne*) also made a mark on mathematics through the answers it brought to bear – and the debates it engendered – on the broader philosophical question that had surrounded the use of these imaginary numbers since the late 16th century, namely their legitimacy in a vision of mathematics dominated by the geometric realism inherited from Antiquity. Argand thus introduced these "impossible numbers" into the framework of this realism and triggered a debate about the necessity thereof, which is not without significance in the progress of this science: those defending the abandonment of this realist

1. This was therefore a second revised publication (1813) of the text of 1806.
2. *Réflexions sur la nouvelle théorie des imaginaires*, *Annales de Gergonne*, Tome V, 1814, pp. 197–209. It is worth recalling that the fundamental theorem of algebra (also known as d'Alembert's or Gauss' theorem) stipulates the existence of n complex roots (distinct or not) for any polynomial of a degree equal to n, and therefore the possibility of factorising it into the product of polynomials of degree 1.
3. Jacques Frédéric Français (1775–1833), a soldier and mathematician, like his brother François Français (1768–1810), was a student at the École Polytechnique (1797). At the time that concerns us here (1813), he was a commander and professor of military tactics at the engineering school in Metz.

dogma in favour of a recognition of the efficiency of algebra alone – of whom the spokesman was F.-J. Servois – attempted to deny the utility of Argand's work, and the exchanges that occurred concerning these two visions of mathematics are highly illuminating for historians of mathematics of the early 19th century. It is therefore of great interest to us to read the two texts by Servois and Argand (as well as the accompanying notes by Gergonne) in the *Annales de mathématiques pures et appliquées*:

- *Lettre de M. Servois sur la théorie des imaginaires*, Tome IV, pp. 228–235
- *Réflexions sur la nouvelle théorie des imaginaires*, Argand, Tome V, pp. 197–209

Lastly, and unlike Wessel's text, Argand's work offers veritable demonstrations based on the geometric figure: his article is supplemented by many drawings, and for this reason adduces more evidence for the concept of the "geometric representation" of complex numbers.

The final arguments behind our choice are borrowed from H. Valentiner, who wrote the preface to the 1897 edition of Wessel's work:

On a cru jusqu'ici que, dans son *Essai sur une manière de représenter des quantités imaginaires*, Paris 1806, Argand était le véritable fondateur de la représentation moderne des nombres complexes comme lignes ayant une direction déterminée. Cependant il est démontré que, dès 1799, Gauss a eu la même idée, et déjà vers la fin du 17e siècle Wallis a essayé de donner aux nombres imaginaires une signification réelle (*A treatise of Algebra*, London 1685, chap. 66-69). Il est donc possible de reporter la première trace de la théorie en question à un temps beaucoup plus reculé que celui qu'on avait supposé jusqu'ici.

Toutefois le traité d'Argand est, de tous ceux qu'on avait bien remarqués jusqu'ici, celui qui représente le plus complètement la théorie des nombres imaginaires, et qui toujours pour cette raison conserve un intérêt historique;

[Until now it was believed that, in his Essai sur une manière de représenter des quantités imaginaires*, Paris 1806, Argand was the true founder of the modern representation of complex numbers such as lines with a definite direction. However, it has been demonstrated that Gauss had the same idea in 1799, and even as early as the 17th century Wallis tried to ascribe genuine meaning to imaginary numbers (*A Treatise of Algebra*, London, 1685, chap. 66–69). It is therefore possible to plot the first trace of the theory in question to a much earlier time than hitherto supposed.*

However, of all those that have achieved some renown, Argand's treatise is the one that represents the theory of imaginary numbers the most completely, and for that reason retains historical interest]

ARGAND'S *ESSAI*

The first part of Argand's *Essai* sets out the mathematical advances due to this concept that is the geometrical representation of imaginary numbers. The rest of his text is a series of demonstrations of the already known properties on trigonometric lines, which are themselves interesting owing to the use of his new tool.

As we have already mentioned, in 1813 Argand returned to the ideas developed in his *Essai* of 1806. The latter had indeed been brought to the attention of J. F. Français, who was then teaching at the Imperial School of Artillery and Engineering in Metz, who delivered his own version in an article entitled "Nouveaux principes de géométrique de position et interprétation géométrique des symboles imaginaires", *Annales de Gergonne*, Tome IV, September 1813, pp. 61–72. This provoked a reaction from Argand ("Essai sur une manière de représenter les quantités imaginaires, dans les constructions géométriques", *Annales de Gergonne*, Tome IV, November 1813, pp. 133–147), followed by the recognition by Français that Argand had devised these concepts before he had. Indeed, the two men continued to exchange news about their advances on the subject over the years 1813–1815.

Argand's *Essai* of 1806 cannot be studied without drawing parallels with the author's revised version of 1813. The following lines are thus a synthesis of the comparative studies of these two texts.

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Argand starts from the widely accepted idea that a ratio between two quantities "of a kind yielding negative values" comprises two notions: 1°) that of the numerical relation between the absolute values; and 2°) that of the relation of the *directions* (also known as *senses*), a relation either of *identity* or *opposition*. Ultimately, the relation $\frac{+a}{-b} = \frac{-ma}{+mb}$ states both the fact that $\frac{a}{b} = \frac{ma}{mb}$ and the idea that "the quantity +a is, relative to the direction of the quantity -b, what the direction of -ma is relative to the direction of +mb", which can be expressed more simply as $\frac{+1}{-1} = \frac{-1}{+1}$ (A).

To document these two concepts, he provides a perfect definition of **algebraic measure**, without using the expression that is used today, and moreover does so by introducing the notation we now know: $\overline{AB}$. Thus, the line AB, representing a number "considered in its absolute magnitude", defines two opposite directed lines, $\overline{AB}$ and $\overline{BA}$, of the same position. In the figure below, for example, if KA corresponds to +1, proposition (A) above translates as: "KA is to KI what KI is to KA".

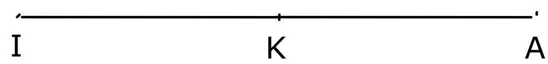

Figure 2

This notion of the directed line $\overline{AB}$ is stated in the following terms by Argand (§ 6, p. 11):

> *They will be called* lines having direction *or simply* directed lines*. They will be thus distinguished from* absolute *lines whose length only is considered without regard to direction.*

Argand's idea was thus to extrapolate these concepts of *absolute magnitude* and *direction* to imaginary numbers, which would naturally lead to what we call the modulus and argument of a complex number.

Argand considers the proportion $\frac{+1}{x} = \frac{x}{-1}$ and observes that no positive or negative number (he is of course talking about real numbers) is appropriate: if the quantity being sought exists, it is therefore *imaginary*.

Argand thus has the idea of noting the unit taken in the direction d as 1_d, and in the same manner seeks a direction d such that the positive direction is to d what the latter is to the negative direction, which, by "generalising" (A), is written as:

$$\frac{+1}{1_d} = \frac{1_d}{-1} \quad \text{(B)}$$

In his eyes, proportion (B) actually contains two very different identifications of signification and scope: a proportion of a numerical nature ($\frac{+1}{-1} = \frac{-1}{+1}$), and "a proportion or similitude of relations of direction analogous to that of the proportion

(A)". Argand adds: "and, since one acknowledges the truth of the latter, one cannot refuse to also recognise the legitimacy of proportion (B)". This was a risky generalisation, but it was one that would prove correct and which enabled a rapid advance in the representation of complex numbers.

The fundamental principle behind his theory is based on the idea of proportions between directed lines:

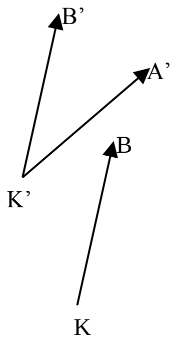

Figure 3

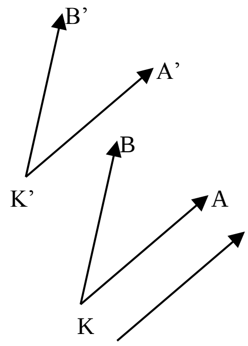

Figure 4

In Figure 4, the direction of KA is to that of KE as KE is to the direction of KI. Argand writes this as $\frac{+1\,(KA)}{1_d(KE)} = \frac{1_d(KE)}{-1\,(KI)}$ and once again sees it as a double proportion, i.e. a numerical proportion and a similitude of relations of direction:

> like manner by $\overline{KE}$. For the direction of $\overline{KA}$ is to that of $\overline{KE}$ as is the latter to that of $\overline{KI}$. Moreover we see that this same condition is equally met by $\overline{KN}$, as well as by $\overline{KE}$, these two last quantities being related to each other as $+1$ and -1. They are, therefore, what is ordinarily expressed by $+\sqrt{-1}$, and $-\sqrt{-1}$. In an analogous manner we [4]

KE thus becomes the direction of the pure imaginary numbers (that of i), with KA and KI being those on which he based, by analogy, his construction of real positive and negative numbers.

4. Argand, *Imaginary Quantities: Their Geometrical Representation*, trans. A. S. Hardy, New York, D. Van Nostrand, 1881, available online here (PDF).

In a manner analogous to this relation defined in Fig. 4, in Fig. 3 we have: $\frac{KA}{KB}=\frac{K'A'}{K'B'}$ in direction, *disregarding the absolute magnitudes*, since the angles are equal. The analogy he uses thus leads him to consider the lines both in terms of their *direction* and their *position*: this is the birth of the complex plane.

> primitive unit $\overline{KA}$, it is seen, that every line parallel to the primitive direction is expressed by a real number, that those perpendicular to it are expressed by imaginaries of the form $\pm a\sqrt{-1}$, and, finally, that those having other directions are of the form $\pm a \pm b\sqrt{-1}$, and are composed of a real and imaginary part. But these lines are quanti-

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We now see the result towards which Argand is heading: under the misleading guise of the term “proportion”, he has constructed a hybrid being borrowed from both algebra and geometry, which enables him to retain the idea of proportion and the underlying notion of equality, but which eludes the problem of relation of order: where a proportion between real numbers would enable a comparison, the same is not true of imaginary numbers; there is a lack of permanence that is not identified. So long as one confined oneself to the operational rules on complex numbers, as had been the case since the 16th century, their set included that of the real numbers and furthermore fulfilled the principle of permanence, that is to say that the operational rules valid for real numbers remained valid (hence the name permanence) in the set of imaginary numbers: commutativity, distributivity, the role of zero, etc. But whereas the geometric representation of the real numbers on a straight line made it possible to report a relation of order, a “sorting” of those numbers, the same does not hold for directed lines and imaginary numbers (which are not “ordered”). There is therefore a loss of permanence in the link between the figure and the ordering.

Yet in losing the latter, Argand gains something else: complex numbers, to all appearances, have geometric legitimacy. He pursues his work by introducing notations that are used today with occasionally more restricted significations.

The *direction* of AB will be noted $\overline{AB}$ or $\overline{BA}$, depending on whether the *directed line* is directed from A towards B or from B towards A, these two directed lines sharing the *position* that *collectively designates two opposite directions*: this is a classification of the directed lines by their positions, an idea that contains the notion of equivalence classes in embryonic form; it is another way of formulating the idea that was expressed above in the equality of the relations in the direction of KA, KB, K'A', K'B'.

In fact, a third characteristic magnitude is named but does not receive the emphasis it deserves: *absolute magnitude*. At the time this was considered natural, since the questions relating to the definition of the concept of distance had not yet been posed. Yet, besides this remark, we can see that here Argand provided almost the first definition of the concept of a vector:[5] we will confirm this by a detailed study of the operating modes on his *directed lines*, defined as illustrations of the operations on *imaginary numbers*.

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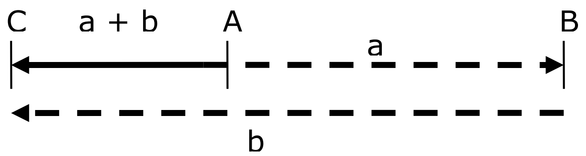
Argand defines the sum of two directed lines as we define the sum of two vectors. And, here again, *analogy* plays a preponderant role in the extension of a concept: starting from the geometric illustration of the addition of relative numbers, he extends this "by reasoning by analogy", by suggesting the addition of *directed lines* according to the same principle:

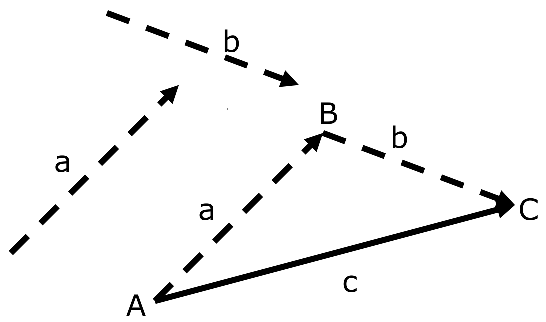


The relative number a + b is "represented" by AC

Figure 5

The notations with arrow-marked segments are not Argand's, but it is clear (Fig. 6) that this is indeed the addition of two vectors on a vectorial plane, as was the case for the relative numbers (Fig. 5) in what is known as the Chasles relation on algebraic measures: $\overline{AC} = \overline{AB} + \overline{BC}$

5. Its *direction* is what we now call *sense*, its *absolute magnitude* its *norm*, and its *position* its *direction*. Our secondary school pupils define the vector on the basis of these three characteristics, with the "abstract" definition (via the concept of vector space) being postponed until a more advanced level of study.



The sum of the directed lines a and b is "represented" by the directed line c = AC

Figure 6

But Argand further explores what would later become the "vectorial field". He perfectly defines the notion of the decomposition of a vector over a basis and the link that exists between an orthonormal reference point and the ordered pair (1, $\sqrt{-1}$), and indeed goes even further:

> *A line in the given direction* $\overline{\mathrm{KP}}$ *can be decomposed into two parts belonging to the given* *<u>positions</u>* *KA and KB. For that, it suffices to draw, onto KB, KA, lines PM, PN parallel to KA, KB, and we will have:* $\overline{\mathrm{KP}} = \overline{\mathrm{KM}} + \overline{\mathrm{MP}} = \overline{\mathrm{KN}} + \overline{\mathrm{NP}}$ *; but as we have* $\overline{\mathrm{KM}} = \overline{\mathrm{NP}}$ *and* $\overline{\mathrm{KN}} = \overline{\mathrm{MP}}$*, and as moreover there are only these two ways to operate the proposed decomposition, one must conclude, in general, that if, having* $\overline{\mathrm{A}} + \overline{\mathrm{B}} = \overline{\mathrm{A}}' + \overline{\mathrm{B}}'$*, A, A′ have the same direction a, B, B′ have the same direction b, with a and b not belonging to the same position, one must also have:* $\overline{\mathrm{A}} = \overline{\mathrm{A}}'$ and $\overline{\mathrm{B}} = \overline{\mathrm{B}}'$
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> *This partition frequently occurs when one of the positions is that of* ± 1 *and the other the perpendicular position, which amounts to the separation of the real number and the imaginary number.* (Annales, *Tome IV, p. 138)*

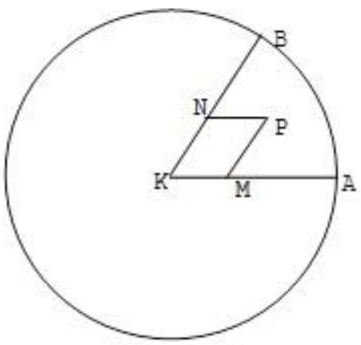

Figure 7

Argand finally states here that:

> *Given two vectors* $\overline{\mathrm{KA}}$ *and* $\overline{\mathrm{KB}}$ [he uses the phrases "directed lines"; moreover, today we would write them with arrows above them] *not having*

the same direction [he says position], *then any vector* $\overline{\mathrm{KP}}$ *can be decomposed in two ways as the sum of two vectors of the same directions as* $\overline{\mathrm{KA}}$ *and* $\overline{\mathrm{KB}}$*. If one takes into account the order of the decomposition, there is therefore uniqueness.*

There is no doubt that this is the decomposition of a vector over a *basis*: though he does not employ this term in his article in the *Annales de Gergonne*, he had used it in his *Essai* of 1806. The notion of the vector was of course present in many older works: the rule of the parallelogram had long featured in the study of the composition of certain movements, such as the composition of the forces applied to the same point, in Newton for example.

But Argand takes this notion to another level of abstraction: his *directed lines* **are** abstract entities. They are connected to the representants of the figure only by the underlying relation of equipollence: the *position* is not the description of a perceptible quality on the figure; it is rather a concept capable of embracing an infinite diversity within the same whole. And the properties of decomposition and uniqueness seen above are valid, as he well knows, only for the abstract entity and not for its planar representations: a vector is unique as a class of equipollence, but its representations in the plane are infinite in number. This relation of equipollence, hinted at and used by Argand (and Wessel before him), would be defined and developed from 1832 onwards, independently of the results of their work, by the Italian Giusto Bellavitis.[6] It was he who gave his name to this mathematical concept, and explored it in greater depth in numerous publications, particularly in the *Annali delle scienze del regno Lombardo – Veneto* from 1835 to 1838.[7]

The notion of equipollence today

The notion of equipollence has not changed since it was defined by Bellavitis (see above) in the mid 19th century. Equipollence was later studied and considered as a relation of equivalence, and it was thus possible to define vectors more precisely (see panel below). The notion of vector space appeared much later: it is now possible to do without equipollence by defining a vector simply as an element in a vector space. But that's another story…

6. Giusto Bellavitis (1803–1880), an Italian mathematician, defined the concept of equipollence and helped in the modern formulation of the concept of the vector between 1832 and 1845. When his contribution to mathematics (equipollence, vectors, quaternions) is mentioned, the earlier work of Argand and Français in the *Annales de Gergonne* often tends to be overlooked.
7. An exhaustive list of Bellavitis' work in this area can be found in Elie Cartan, 1953, pp. 344–345.

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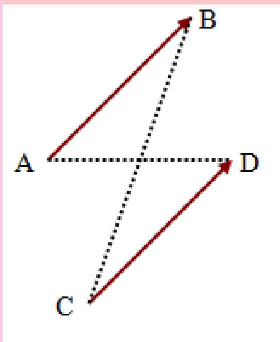
Two bipoints (or pairs of points) (A,B) and (C,D) are called equipollent if [AD] and [BC] have the same middle, i.e. if the quadrilateral ABDC (in this order) is a parallelogram (in the broad sense, since the definition is also valid for aligned points, and thus for a "flattened" quadrilateral).
The set of all the bipoints equipollent to (A,B) thus defines a same and new mathematical object (in fact, a class of equivalence of this relation of equipollence): the *vector* $\overrightarrow{AB}$. This is a definition of what we call a vector, which is therefore an infinite set of bipoints, equipollent among themselves, and of which only the "representants" can be shown ("drawn").
We owe this notion to Bellavitis, even though he did not speak in terms of vectors but rather of "equipollent lines", which are in fact reminiscent of Argand's "directed lines".



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Having thus defined the addition of directed lines, which enables him to perfectly represent the addition of imaginary numbers, Argand turns to a way of representing their product: he thus describes a way of multiplying directed lines that respects the rules of arithmetic. In his *Essai* of 1806, using first the unit circle, he defines the construction of the "product" of the directed lines $\overline{KB} \times \overline{KC}$ in the following manner:

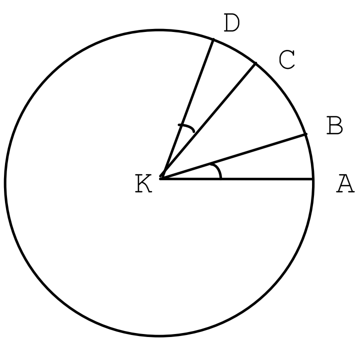

Figure 8

Construct the angle $\overline{CKD} = \overline{AKB}$. [8]

From what was said in No. 4, Note I, we have $\overline{KA} : \overline{KB} :: \overline{KC} : \overline{KD}$, whence $\overline{KA} \times \overline{KD} = \overline{KB} \times \overline{KC}$; but $\overline{KA} = +1$, hence $\overline{KB} \times \overline{KC} = \overline{KD}$. Therefore, to construct the product of two directed radii, lay off, from the origin of arcs, the sum of the arcs corresponding to each radius, and the extremity of the arc thus laid off will determine the position of the radius of the product; this, as before, is logarithmic multiplication. It is un- [9]

Since this concerns representations of complex numbers of modulus 1, he geometrically states the fact that the argument of the product of two of these numbers is equal to the sum of their arguments. He himself makes note of this with the expression "logarithmic multiplication".

The modulus and argument of a complex number

For a complex number z = a + ib, in modern notation:

- the module is defined by $|z| = \sqrt{a^2 + b^2}$, and corresponds to the length of the corresponding vector
- the argument is defined by $\theta = \mathrm{Arctg}\left(\frac{b}{a}\right)$ and corresponds to the angle subtending the corresponding vector in relation to the axis of the abscissas

If we rewrite the above paragraph from Argand in modern complex notation, we write:

$$\overline{KA} = 1,\ \overline{KB} = \cos\beta + i\sin\beta,\ \overline{KC} = \cos\gamma + i\sin\gamma$$

Multiplying, we have:

$$\overline{KB} \times \overline{KC} = (\cos\beta\cos\gamma - \sin\beta\sin\gamma) + i\,(\sin\beta\cos\gamma + \cos\beta\sin\gamma)$$

$$\overline{KD} = \overline{KB} \times \overline{KC} = \cos(\beta + \gamma) + i\sin(\beta + \gamma)$$

Following the above definition, the argument of KB (the angle formed by the radius KB with the axis of the abscissas KA, Fig. 8) is ß, and the argument of KC is γ; as for the argument of KD,

8. He in fact employs the notation of the time, namely savoir KA:KB::KC:KD, with :: symbolising equality.
9. "Directed radii" = KB and KC; "radius of the product" = KD.

according to the last formula, it is equal to ß + γ. The argument of the product of KB and KC is equal to the sum of the arguments of KB and KC. This is what Argand (p. 41) calls a "logarithmic multiplication", that is to say, the transformation of a product into a sum.
Among the many applications of his method that he gives at the end of the text, Argand demonstrates, with his notations, that:

$$\cos na \sim \sin na = (\cos a \sim \sin a)^n$$

This identity is well known in modern notation as the Moivre formula (1707):

$$(\cos na + i \sin na) = (\cos a + i \sin a)^n$$

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Argand concludes his construction with directed lines *that are not units* (i.e. with complex numbers of given moduli), stating that if one wishes to effect the product of $m\overline{KB}$ by $n\overline{KC}$, one simply has to construct the directed line $mn\overline{KD}$: this time, he geometrically expresses the fact that the modulus of the product of two complex numbers is equal to the product of their moduli.

We won't go into further detail about the consequences of these two definitions, which Argand develops equally well in his two major essays. Let's simply point out that he **perfectly** states a great many of the properties of complex numbers and trigonometric lines, in both their algebraic expression and their geometric representation. The consequences would be immense: trigonometry formulas became specific cases of de Moivre's formula underlying these writings, sums of series would be found, and in the *Annales de Gergonne*[10] Argand even gave a demonstration of d'Alembert's theorem. This demonstration was far from satisfying since it constructed a sequence of complex numbers that is supposed to decrease towards 0, this convergence not being proven.

THE RULE OF LINES

As we have seen, Argand formalises the notion of the vector in modern terminology and ultimately establishes the correspondence between the vector space associated with a plane and the set of imaginary numbers. He went even further in his *Essai* of 1806: putting forward symbols specifically devoted to

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10. *Réflexions sur la nouvelle théorie des imaginaires, suivies d'une application à la démonstration d'un théorème d'analise*, Tome V, p. 197, January 1815.

imaginary numbers, he ascribes to them the operational rules that ensure the permanence of the hitherto acknowledged operations, and even draws on the congruence, as the following extract shows:

of $\dot{a}$; but really $\sqrt{-1}$, in $a\sqrt{-1}$, is no more a factor than is $+1$ in $+a$, or -1 in $-a$. Now we do not write $+ 1.a, - 1.a$, but simply $+ a, - a$, and the sign which precedes a itself indicates what kind of a unit this number expresses. We may then apply a similar method to imaginary quantities, writing for example $\sim a$ and $\not\sim a$ instead of $+a\sqrt{-1}$ and $-a\sqrt{-1}$, the signs $\sim$ and $\not\sim$ being reciprocally positive and negative. To multiply these signs, we observe that either multiplied by itself gives $-$, and, consequently, multiplied by each other they give $+$. Moreover, a single rule, applicable to any number of factors, may be established; let every straight line, horizontal or vertical, in the signs to be multiplied, have a value 2, and every curved one a

value 1; we shall have for the four signs the following values:

$$\sim=1, \quad -=2, \quad \not\sim=3, \quad +=4.$$

Then take the sum of the values of all the factors and subtract as many times 4 as is necessary to make the remainder one of the numbers 1, 2, 3, 4; this remainder will be the value of the sign of the product; and so, for division, sub-

These new signs would abridge the notation,* and perhaps render the calculus of imaginaries more convenient, errors of sign being sometimes easily

* The quantity $m \pm n\sqrt{-1}$ being denoted by $m \sim n$, or by $m \not\sim n$, the single sign $\sim$, or $\not\sim$, replacing the four signs $+$, $\sqrt{\ }$, $-$, 1.

made.* We shall employ them in what follows, without implying on that account that they should be adopted. Doubtless to every innovation, even a rational one, there is an intrinsic objection; but no progress would be made if they were rejected, for the only reason that they are contrary to usage, and their trial, at least, is permissible.

* For example, let it be required to multiply $-m\sqrt{-c}$ by $+n\sqrt{-cd}$. The product of the two coefficients is $-mn$; that of the two radicals is $-c\sqrt{d}$; and the final product is $+mnc\sqrt{d}$. In the new notation the two factors are $\sim m\sqrt{c}$, $+n\sqrt{cd}$, or $+m\sqrt{c}$, $\sim n\sqrt{cd}$, and by the rule we at once obtain $+mnc\sqrt{d}$. This advantage—if it be one—would not exist for an experienced calculator, who by a simple inspection of the factors would read the product; but not every one possesses this faculty. [11]

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A fine attempt at abstraction and generalisation for a mathematician who wished to better anchor the imaginary numbers in geometric reality!

But the question remained: even if this "geometric representation" of complex numbers would go on to become established in the first half of the 19th century (thanks to Argand perhaps, but also to Gauss, and later Cauchy), did this attempt meet with success? We will borrow the following conclusion from Elie Cartan:[12]

> *In the theory that concerns us* [Argand's]*, all numbers, real or imaginary, are defined by vectors, situated on a given plane, and having, in this place, a common origin O and being subject to the operations (addition, subtraction, multiplication, division) that one defines through suitable conventions. We define these conventions such that:*
>
> *1°) the operations defined enjoy the same properties as the operations of the algebra of real numbers* [this is the principle of permanence[13] applied to the laws of algebraic calculus]
>
> *2°) in the specific case where the vectors subject to operations are carried by a particular oriented straight line passing through O (and which we will name the real axis), these operations are identical to those that have been*

11. Argand substitutes the rule of signs with a rule of lines +, −, ~ , ~ with | crossing out the ~. Its application to the example given by Argand gives a multiplication of by , i.e. an addition of the figures 1 and 3 in the rule of lines, which gives 4, and therefore a + sign.
12. Elie Cartan, *Œuvres complètes*, Gauthier Villars, Paris, 1953, p. 340.
13. See above for more on the principle of permanence.

defined in the metric theory of real numbers [this is the principle of permanence applied to the laws of vector calculus].

Finally, Argand reproduces – by analogy, in the geometric field, and again respecting the principle of permanence reiterated by Cartan – the technique of extension that presided over the establishment of imaginary numbers and the operations concerning them in the field of algebra. He shifts the problem: though there is admittedly a figure that "translates" the properties of imaginary numbers, these do not always have a "reality", in the sense that they are still not related to an even imperfect projection in reality, via even an approximate figure or object.[14] He demonstrates the adequacy of his construction in the plane with that already existing on the straight line, and contents himself, as his contemporaries would later on, with this "parallel". Imaginary numbers would retain a certain air of mystery. As C. F. Gauss (1777–1855) put it poetically:

> *The true meaning of √-1 reveals itself vividly before my soul, but it will be very difficult to express it in words, which can give only an image suspended in the air.* (Letter to Peter Hanson, 1825)

The truly innovative aspect of Argand's work became apparent at a later date, in the underlying concepts he uses, as we saw above (vectors, "natural" isomorphisms, congruences, etc.), and which would shortly unify the two parallel fields he had just bridged. To once again cite Cartan:

> *Nor does one* [Argand] *propose to simply interpret the imaginary solutions of certain equations of geometric origin, as J. Wallis, H. Kühn, and A. Q. Buée had attempted: such an interpretation indeed presupposes, at least in theory, the legitimacy of calculus with imaginary symbols. The geometric theory of complex numbers is a natural generalisation of the metric theory of irrational numbers in which each real number is defined by a segment of a given oriented straight line.*[15]

The cautious Argand, expecting hostile reactions (like that of Servois), replies in the following terms:

> *The theory of which we have just given an overview may be considered from a point of view apt to set aside the obscure in what it presents, and which seems to be the primary aim, namely: to establish new notions on imaginary quantities. Indeed, putting to one side the question of whether these notions*

14. For example, for the irrational number √2 a right-angled isosceles triangle of side 1, or for the transcendent number ∏ an approximation of the circle by regular polygons: thus, there is something in reality that "de-idealises" these numbers. Nothing of the sort exists for the imaginary number i, despite Argand's work: the problem remains entirely unresolved simply because, in the sense of the "geometric realism" inherited from the Ancients, √-1 has no legitimacy and cannot acquire any.
15. Ibid.

are true or false, we may restrict ourselves to viewing this theory as a means of research, to adopt the lines in direction only as ***signs*** *of the real or imaginary quantities, and to see, in the usage to which we have put them, only the* ***simple employment of a particular notation****. For that, it suffices to start by demonstrating, through the first theorems of trigonometry, the rules of multiplication and addition given above; the applications will follow, and all that will remain is to examine the question of didactics. And if the employment of this notation were to be advantageous? And if it were to open up shorter and easier paths to demonstrate certain truths? That is what fact alone can decide.*[16]

And indeed, this is what the history of mathematics, its progress, applications and teaching have amply demonstrated ever since.

■

(January 2009)

(Translation by Helen Tomlinson, published April 2017)

16. *Essai sur une manière de représenter les quantités imaginaires, dans les constructions géométriques*, *Annales*, Tome IV, November 1813, pp. 133–147.